

# ON THE GUIGNARD CONSTRAINT QUALIFICATION FOR MATHEMATICAL PROGRAMS WITH EQUILIBRIUM CONSTRAINTS

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January 9, 2004

**Abstract.** We recapitulate the well known fact that most of the standard constraint qualifications are violated for mathematical programs with equilibrium constraints (MPECs). We go on to show that the Abadie constraint qualification is only satisfied in fairly restrictive circumstances. In order to avoid this problem, we fall back on the Guignard constraint qualification. We examine its general properties and clarify the position it occupies in the context of MPECs. We show that strong stationarity is a necessary optimality condition under Guignard CQ. Also, we present several sufficient conditions for Guignard CQ, showing that it is usually satisfied for MPECs.

**Key Words.** Mathematical programs with equilibrium constraints, Guignard constraint qualification, strongly stationary, constraint qualifications

# 1 Introduction

Consider the constrained optimization problem

$$\begin{aligned} \min \quad & f(z) \\ \text{s.t.} \quad & g(z) \leq 0, \quad h(z) = 0, \\ & G(z) \geq 0, \quad H(z) \geq 0, \quad G(z)^T H(z) = 0, \end{aligned} \tag{1}$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $h : \mathbb{R}^n \rightarrow \mathbb{R}^p$ ,  $G : \mathbb{R}^n \rightarrow \mathbb{R}^l$ , and  $H : \mathbb{R}^n \rightarrow \mathbb{R}^l$  are continuously differentiable functions. Programs of type (1) are known as mathematical programs with complementarity constraints. Alternatively, (1) is called a mathematical program with equilibrium constraints, or MPEC for short. Usually an MPEC is an optimization problem with a variational inequality as a constraint. Under certain circumstances, however, MPECs can be written in the form (1) (see [12] for an extensive discussion on this). Since MPEC is a nicer acronym than MPCC, we will stick to calling (1) an MPEC.

Of prime importance to the theory of any class of nonlinear programs are optimality conditions. In this paper we consider first order conditions as they are derived from Karush-Kuhn-Tucker (or KKT) points for standard nonlinear programs.

As is well known, such KKT points are necessary optimality conditions only in the presence of constraint qualifications. Unfortunately, most of the usual constraint qualifications (called CQs hereafter) from standard nonlinear programming are known to be violated for MPECs (see, e.g., [2]). Even the Abadie CQ, typically the weakest CQ considered for standard nonlinear programs, does not hold in general. In fact, some fairly restrictive assumptions are needed in order to guarantee that the Abadie CQ holds for the MPEC (1), see Section 2 and the corresponding discussion by Pang and Fukushima [15].

A constraint qualification still weaker than the Abadie CQ is the Guignard CQ, see [8], and the overview article [16] for a finite dimensional formulation of the Guignard CQ and how it stands in relation to other classic constraint qualifications. It seems that this CQ has been widely overlooked in the MPEC literature. In fact, we know of no real application of the Guignard CQ. We speculate that this is because for standard nonlinear programs, Abadie CQ is weak enough to be satisfied for most of the interesting cases. Since, as we will note, Abadie CQ is not weak enough in the context of MPECs, we turn to the Guignard CQ. In Section 3 we define it and recap some of its properties and alternate definitions (see, in particular, [1, 20]).

Finally, in Section 4 we show that the Guignard CQ implies some strong first order optimality conditions for the MPEC (1). We also derive some sufficient conditions for the Guignard CQ to be satisfied. The material presented in Section 4 is closely related to the work by Pang and Fukushima [15], where similar statements are derived without, however, referring to the Guignard CQ. Nevertheless, we wish to stress that our work is heavily based on their paper [15].

Although we restrict our sojourn into constraint qualifications for MPECs to a few selected ones, many more constraint qualifications have been proposed, discussed, and shown to yield certain first order conditions. Generalizations of CQs have been discussed in [18] and shown to imply various stationarity concepts. In [15, 18] one stationarity

concept, namely strong stationarity, has been considered in depth. A weaker stationarity concept, M-stationarity, has been the focus of recent theoretical analysis, most notably in [13, 14].

We also note that the different stationarity concepts play an important role in the design of suitable algorithms for the MPEC (1). For a more detailed analysis, the interested reader is referred to the recent papers [3, 19, 9, 6], for example.

A word on notation;  $\mathbb{R}^n$  denotes the  $n$ -dimensional Euclidean space. For  $x \in \mathbb{R}^n$  and  $y \in \mathbb{R}^m$  we simply write  $(x, y)$  for the  $(n+m)$ -dimensional vector  $(x^T, y^T)^T$ . Given a vector  $x \in \mathbb{R}^n$  and a subset  $\delta \subseteq \{1, \dots, n\}$ , we denote by  $x_\delta$  the subvector in  $\mathbb{R}^{|\delta|}$  consisting of all components  $x_i$  with  $i \in \delta$ . Inequalities  $x \geq 0$  for arbitrary vectors  $x \in \mathbb{R}^n$  are defined componentwise. We will make frequent use of the following index sets. For a given feasible point  $z^*$  of the MPEC (1), they are defined as follows:

$$\alpha := \alpha(z^*) := \{i \mid G_i(z^*) = 0, H_i(z^*) > 0\}, \quad (2a)$$

$$\beta := \beta(z^*) := \{i \mid G_i(z^*) = 0, H_i(z^*) = 0\}, \quad (2b)$$

$$\gamma := \gamma(z^*) := \{i \mid G_i(z^*) > 0, H_i(z^*) = 0\}. \quad (2c)$$

The set  $\beta$  is called the *degenerate* set. Furthermore,  $\mathcal{P}(\beta)$  denotes the set of all partitions of  $\beta$ , where a pair  $(\beta_1, \beta_2)$  is a partition of  $\beta$  if  $\beta_1 \cup \beta_2 = \beta$  and  $\beta_1 \cap \beta_2 = \emptyset$ .

Since cones play a central role in this paper, we wish to clarify what we mean by cone. A set  $\mathcal{C}$  is called a cone if  $\lambda x \in \mathcal{C}$  for all  $x \in \mathcal{C}$  and  $\lambda \geq 0$ . It is in this last point that the literature is somewhat ambiguous. It is common to define a cone with the strict inequality for  $\lambda$ . In our definition, however, a nonempty cone always contains 0.

## 2 Standard Constraint Qualifications

As mentioned in the introduction, it is a well-known fact that most of the familiar constraint qualifications known for standard nonlinear programs do not hold for MPECs of type (1). See, e.g., [2, 23] for more details.

Clearly, the constraints of (1) are not affine (neglecting the uninteresting case when either  $G_i(\cdot)$  or  $H_i(\cdot)$  is constant for every  $i = 1, \dots, l$ ). In particular, the MPEC (1) is not a convex program, rendering the application of Slater CQ useless. Additionally, Slater CQ can be easily verified never to hold for any feasible point of (1).

We also note that the Mangasarian-Fromovitz constraint qualification (referred to as MFCQ in the following) is violated for every feasible point of (1), see, e.g., [23]. Since the linear independence constraint qualification implies MFCQ, it also is never satisfied.

We now turn our attention to the Abadie constraint qualification (referred to as ACQ in the following). Recall that it is said to hold in a feasible point  $z^*$  if

$$\mathcal{T}(z^*) = \mathcal{T}^{lin}(z^*), \quad (3)$$

where  $\mathcal{T}(z^*)$  is the tangent cone and  $\mathcal{T}^{lin}(z^*)$  is the linearized tangent cone of the MPEC (1) in the point  $z^*$ . As a quick reminder, if  $\mathcal{Z}$  is the feasible region of the MPEC (1), the

tangent cone  $\mathcal{T}(z^*)$  is defined as follows:

$$\mathcal{T}(z^*) := \left\{ d \in \mathbb{R}^n \mid \exists \{z^k\} \subset \mathcal{Z}, \exists t_k \searrow 0 : z^k \rightarrow z^* \text{ and } \frac{z^k - z^*}{t_k} \rightarrow d \right\}, \quad (4)$$

while it is easy to see that the linearized tangent cone  $\mathcal{T}^{lin}(z^*)$  of (1) can be expressed as

$$\begin{aligned} \mathcal{T}^{lin}(z^*) = \{d \in \mathbb{R}^n \mid & \nabla g_i(z^*)^T d \leq 0 & \forall i \in \mathcal{I}_g := \{i \mid g_i(z^*) = 0\}, \\ & \nabla h_i(z^*)^T d = 0 & \forall i = 1, \dots, p, \\ & \nabla G_i(z^*)^T d = 0 & \forall i \in \alpha, \\ & \nabla H_i(z^*)^T d = 0 & \forall i \in \gamma, \\ & \nabla G_i(z^*)^T d \geq 0 & \forall i \in \beta, \\ & \nabla H_i(z^*)^T d \geq 0 & \forall i \in \beta\}. \end{aligned} \quad (5)$$

Note that the inclusion  $\mathcal{T}(z^*) \subseteq \mathcal{T}^{lin}(z^*)$  always holds and that  $\mathcal{T}(z^*)$  is closed but not necessarily convex, while  $\mathcal{T}^{lin}(z^*)$  is polyhedral and hence both closed and convex.

Before continuing, we need to introduce a program derived from the MPEC (1) for an arbitrary feasible point  $z^*$ : Given a partition  $(\beta_1, \beta_2) \in \mathcal{P}(\beta)$ , let  $\text{NLP}_*(\beta_1, \beta_2)$  denote the following nonlinear program:

$$\begin{aligned} \min & f(z) \\ \text{s.t.} & g(z) \leq 0, \quad h(z) = 0, \\ & G_{\alpha \cup \beta_1}(z) = 0, \quad H_{\alpha \cup \beta_1}(z) \geq 0, \\ & G_{\gamma \cup \beta_2}(z) \geq 0, \quad H_{\gamma \cup \beta_2}(z) = 0. \end{aligned} \quad (6)$$

Note that the program  $\text{NLP}_*(\beta_1, \beta_2)$  depends on the vector  $z^*$ .

In the following we will repeatedly need the following assumption (A1). Note that it coincides with the first part of assumption (A1) found in [15].

**(A1)** For every partition  $(\beta_1, \beta_2) \in \mathcal{P}(\beta)$ , the Abadie CQ holds for  $\text{NLP}_*(\beta_1, \beta_2)$  in  $z^*$ , i.e.

$$\mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}(z^*) = \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{lin}(z^*),$$

where  $\mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}(z^*)$  is the tangent cone of the program  $\text{NLP}_*(\beta_1, \beta_2)$  and  $\mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{lin}(z^*)$  is the corresponding linearized tangent cone:

$$\begin{aligned} \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{lin}(z^*) = \{d \in \mathbb{R}^n \mid & \nabla g_i(z^*)^T d \leq 0, & \forall i \in \mathcal{I}_g, \\ & \nabla h_i(z^*)^T d = 0, & \forall i = 1, \dots, p, \\ & \nabla G_i(z^*)^T d = 0, & \forall i \in \alpha \cup \beta_1, \\ & \nabla H_i(z^*)^T d = 0, & \forall i \in \gamma \cup \beta_2, \\ & \nabla G_i(z^*)^T d \geq 0, & \forall i \in \beta_2, \\ & \nabla H_i(z^*)^T d \geq 0, & \forall i \in \beta_1\}. \end{aligned} \quad (7)$$

Extending Proposition 3 from [15] trivially, it is possible to prove the following characterization of Abadie CQ for the MPEC (1).

**Proposition 2.1** *Let  $z^*$  be a feasible point of the MPEC (1) and assume that (A1) holds. Then the following statements are equivalent:*

- (a) *the Abadie constraint qualification holds in  $z^*$ ;*
- (b) *there exists a partition  $(\hat{\beta}_1, \hat{\beta}_2) \in \mathcal{P}(\beta)$  such that  $\mathcal{T}(z^*) = \mathcal{T}_{NLP_*(\hat{\beta}_1, \hat{\beta}_2)}(z^*)$ ;*
- (c) *there exists a partition  $(\hat{\beta}_1, \hat{\beta}_2) \in \mathcal{P}(\beta)$  such that  $\mathcal{T}_{NLP_*(\beta_1, \beta_2)}(z^*) \subseteq \mathcal{T}_{NLP_*(\hat{\beta}_1, \hat{\beta}_2)}(z^*)$  for all  $(\beta_1, \beta_2) \in \mathcal{P}(\beta)$ .*

**Proof.** (a) $\Rightarrow$ (b) Since Abadie CQ holds,  $\mathcal{T}(z^*) = \mathcal{T}^{lin}(z^*)$ . Hence,  $\mathcal{T}(z^*)$  is polyhedral and [15, Proposition 3] may be applied to yield the implication.

(b) $\Rightarrow$ (a) Because (A1) holds, we have  $\mathcal{T}(z^*) = \mathcal{T}_{NLP_*(\hat{\beta}_1, \hat{\beta}_2)}^{lin}(z^*)$ , and hence  $\mathcal{T}(z^*)$  is polyhedral. Consequently,  $\mathcal{T}(z^*)$  is generated by linear constraints and is therefore equal to its linearization  $\mathcal{T}^{lin}(z^*)$ , i.e. (a) holds.

(b) $\Leftrightarrow$ (c) It is easy to verify that

$$\mathcal{T}(z^*) = \bigcup_{(\beta_1, \beta_2) \in \mathcal{P}(\beta)} \mathcal{T}_{NLP_*(\beta_1, \beta_2)}(z^*), \quad (8)$$

(see also [4, Lemma 3.1] or [15]). The equivalence of (b) and (c) follows immediately from (8) (note that assumption (A1) is not needed for this equivalence).  $\square$

Geometrically, Proposition 2.1 can be interpreted as follows: While  $\mathcal{T}(z^*)$  is equal to a finite union of (in the presence of (A1)) polyhedral cones (see (8)), the Abadie CQ holds if and only if there is at least one “big” tangent cone in the union (8) which contains all the other tangent cones. It is not difficult to find counterexamples, however, where this is not satisfied, see, e.g., [5, 18].

### 3 Guignard Constraint Qualification

We have shown that most of the standard constraint qualifications are violated in every feasible point of the MPEC (1). Although Abadie CQ has a chance of being satisfied, Proposition 2.1 demonstrates that this is true only under very restrictive circumstances.

We would therefore be well advised to consider alternate constraint qualifications, weaker than Abadie CQ. As mentioned in the introduction, one such constraint qualification is the Guignard CQ (referred to as GCQ in the following).

**Definition 3.1** *The Guignard constraint qualification is said to hold in a feasible point  $z^*$  of (1) if the equality*

$$\mathcal{T}^G(z^*) := \overline{\text{conv}(\mathcal{T}(z^*))} = \mathcal{T}^{lin}(z^*) \quad (9)$$

*holds.*

As mentioned before, the inclusion  $\mathcal{T}(z^*) \subseteq \mathcal{T}^{lin}(z^*)$  holds. Something similar does in fact hold for  $\mathcal{T}^G(z^*)$ , as stated in the following lemma.

**Lemma 3.2** *The inclusion*

$$\mathcal{T}^G(z^*) \subseteq \mathcal{T}^{lin}(z^*) \quad (10)$$

*holds.*

**Proof.** Since  $\mathcal{T}(z^*) \subseteq \mathcal{T}^{lin}(z^*)$  holds and  $\mathcal{T}^{lin}(z^*)$  is both convex and closed, (10) follows immediately.  $\square$

We now want to exhibit a property of the cone  $\mathcal{T}^G(z^*)$ . To this end, consider an arbitrary convex cone  $\mathcal{C}$ . Then the set  $\mathcal{C} \cap (-\mathcal{C})$  is called the *lineality space* of  $\mathcal{C}$ . It is easy to verify that the lineality space of  $\mathcal{C}$  can also be expressed as  $\{y \mid \mathcal{C} + y = \mathcal{C}\}$ . For a more general and detailed discussion of the lineality space, see [17].

We now recall Lemma 3.1.6 from [1] and Corollary 9.1.3 from [17] in the following lemma.

**Lemma 3.3** *Let  $\mathcal{C}_1, \dots, \mathcal{C}_m$  be non-empty convex cones in  $\mathbb{R}^n$ . Then the following hold:*

(i)

$$\mathcal{C}_1 + \dots + \mathcal{C}_m = \text{conv}(\mathcal{C}_1 \cup \dots \cup \mathcal{C}_m). \quad (11)$$

(ii) *Additionally, let  $\mathcal{C}_1, \dots, \mathcal{C}_m$  satisfy the following condition: if  $d_i \in \overline{\mathcal{C}_i}$  for  $i = 1, \dots, m$  and  $d_1 + \dots + d_m = 0$ , then  $d_i$  is in the lineality space of  $\overline{\mathcal{C}_i}$  for  $i = 1, \dots, m$ . Then*

$$\overline{\mathcal{C}_1 + \dots + \mathcal{C}_m} = \overline{\mathcal{C}_1} + \dots + \overline{\mathcal{C}_m}. \quad (12)$$

We will now use the previous lemma to prove the following result.

**Lemma 3.4** *Given a feasible point  $z^*$  of (1), let (A1) hold. Then  $\text{conv}(\mathcal{T}(z^*))$  is closed, i.e. we have*

$$\mathcal{T}^G(z^*) = \overline{\text{conv}(\mathcal{T}(z^*))} = \text{conv}(\mathcal{T}(z^*)).$$

**Proof.** Since, by (A1), Abadie CQ holds for all  $\text{NLP}_*(\beta_1, \beta_2)$  and  $\mathcal{T}(z^*)$  can be written in the form (8), the following holds:

$$\mathcal{T}(z^*) = \bigcup_{(\beta_1, \beta_2) \in \mathcal{P}(\beta)} \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{lin}(z^*). \quad (13)$$

Next, we want to verify that Lemma 3.3 (ii) can be applied to the  $\mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{lin}(z^*)$ . To this end, we recall that  $\mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{lin}(z^*)$  is polyhedral and hence a closed convex cone. Now let  $d_{(\beta_1, \beta_2)} \in \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{lin}(z^*)$  for each  $(\beta_1, \beta_2) \in \mathcal{P}(\beta)$  such that

$$\sum_{(\beta_1, \beta_2) \in \mathcal{P}(\beta)} d_{(\beta_1, \beta_2)} = 0. \quad (14)$$

Multiplying (14) by  $\nabla G_i(z^*)^T$  yields

$$\sum_{(\beta_1, \beta_2) \in \mathcal{P}(\beta)} \nabla G_i(z^*)^T d_{(\beta_1, \beta_2)} = 0 \quad \forall i = 1, \dots, l. \quad (15)$$

Since  $d_{(\beta_1, \beta_2)} \in \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{\text{lin}}(z^*)$ , it holds that  $\nabla G_i(z^*)^T d_{(\beta_1, \beta_2)} \geq 0$  for  $i = 1, \dots, l$ . Hence (15) implies  $\nabla G_i(z^*)^T d_{(\beta_1, \beta_2)} = 0$  for  $i = 1, \dots, l$ .

Taking  $d$  such that  $d + d_{(\beta_1, \beta_2)} \in \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{\text{lin}}(z^*)$ , consider the following:

$$\nabla G_i(z^*)^T d = \nabla G_i(z^*)^T d + \underbrace{\nabla G_i(z^*)^T d_{(\beta_1, \beta_2)}}_{=0} = \nabla G_i(z^*)^T (d + d_{(\beta_1, \beta_2)}) \geq 0, \quad (16)$$

demonstrating that  $d \in \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{\text{lin}}(z^*)$ . Similarly, the above can be shown to hold for  $\nabla g_i(z^*)$ ,  $\nabla h_i(z^*)$ , and  $\nabla H_i(z^*)$ . Conversely, we choose an arbitrary  $d \in \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{\text{lin}}(z^*)$ . Since  $d_{(\beta_1, \beta_2)} \in \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{\text{lin}}(z^*)$  and  $\mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{\text{lin}}(z^*)$  is convex, it holds by standard properties of convex cones that  $d + d_{(\beta_1, \beta_2)} \in \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{\text{lin}}(z^*)$ . Hence we have proven that  $\mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{\text{lin}}(z^*) + d_{(\beta_1, \beta_2)} = \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{\text{lin}}(z^*)$ .

Since  $\mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{\text{lin}}(z^*)$  is convex, Lemma 3.3 (i) and (ii) may be applied.

Now consider the following:

$$\begin{aligned} \overline{\text{conv}(\mathcal{T}(z^*))} &\stackrel{(13)}{=} \overline{\text{conv}\left(\bigcup_{(\beta_1, \beta_2) \in \mathcal{P}(\beta)} \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{\text{lin}}(z^*)\right)} \\ &\stackrel{(11)}{=} \overline{\sum_{(\beta_1, \beta_2) \in \mathcal{P}(\beta)} \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{\text{lin}}(z^*)} \\ &\stackrel{(12)}{=} \overline{\sum_{(\beta_1, \beta_2) \in \mathcal{P}(\beta)} \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{\text{lin}}(z^*)} \\ &= \sum_{(\beta_1, \beta_2) \in \mathcal{P}(\beta)} \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{\text{lin}}(z^*) \\ &\stackrel{(11)}{=} \overline{\text{conv}\left(\bigcup_{(\beta_1, \beta_2) \in \mathcal{P}(\beta)} \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{\text{lin}}(z^*)\right)} \\ &\stackrel{(13)}{=} \overline{\text{conv}(\mathcal{T}(z^*))}. \end{aligned}$$

This concludes the proof.  $\square$

Since the tangent cone  $\mathcal{T}(z^*)$  is always closed, one might hope that the result of Lemma 3.4 held for the convex hull of arbitrary closed cones. The following example, communicated to us by Marco López [11], shows, however, that such a statement is not true in general. Consequently, the statement of Lemma 3.4 is a property of MPECs (under certain assumptions) and is violated in a more general setting.

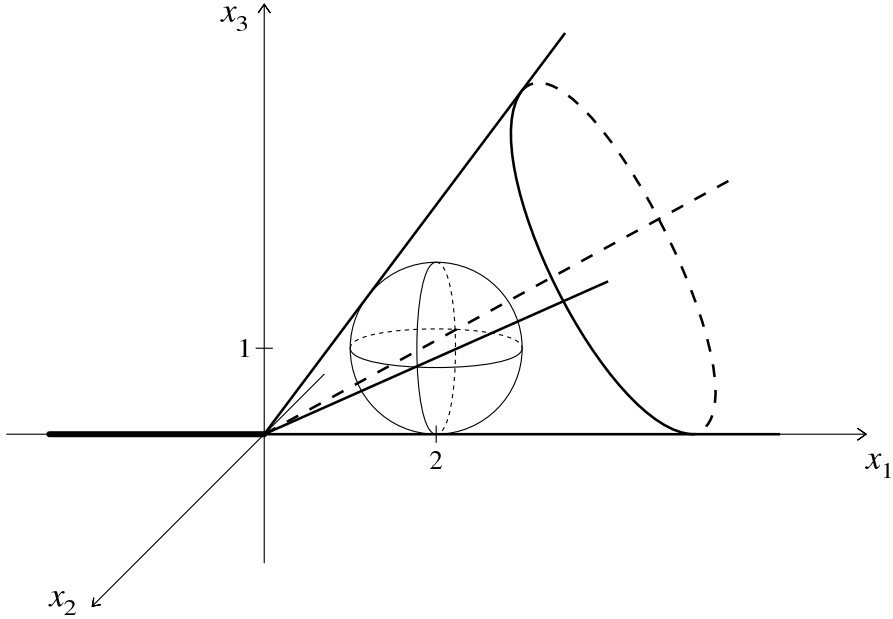


Figure 1: Illustration of  $\text{cone}(\mathcal{S})$  from Example 3.5.

**Example 3.5** Consider the closed, nonconvex cone in  $\mathbb{R}^3$  generated by the set

$$\mathcal{S} = \{(-1, 0, 0)^T\} \cup \{x \in \mathbb{R}^3 \mid \|x - (2, 0, 1)^T\| \leq 1\}.$$

The convex hull of  $\text{cone}(\mathcal{S})$  (see Figure 1) is

$$\text{conv}(\text{cone}(\mathcal{S})) = \{x \in \mathbb{R}^3 \mid x_3 > 0\} \cup \{x \in \mathbb{R}^3 \mid x_2 = x_3 = 0\},$$

which is not closed.

We will now give an equivalent formulation of Guignard CQ, since this will facilitate the proofs of later results. To this end, we must first introduce the concept of the dual cone. Given an arbitrary cone  $\mathcal{C}$ , its *dual cone*  $\mathcal{C}^*$  is defined as follows:

$$\mathcal{C}^* := \{v \in \mathbb{R}^n \mid v^T d \geq 0 \quad \forall d \in \mathcal{C}\}. \quad (17)$$

Note that if a vector  $z^*$  has the property of being B-stationary for the MPEC (1), i.e.

$$\nabla f(z^*)^T d \geq 0 \quad \forall d \in \mathcal{T}(z^*),$$

it is equivalent to stating that  $\nabla f(z^*)$  is in the dual cone of  $\mathcal{T}(z^*)$ , i.e.  $\nabla f(z^*) \in \mathcal{T}(z^*)^*$ .

Cones and their duals have been the subject of extensive research in the past (see, in particular, [1, 17]). In the following Lemma we collect some useful information about the dual cone. Note that in the literature, commonly the *polar cone* is considered, which is simply the negative of the dual cone.



**Lemma 3.6** *Let  $\mathcal{C}$  and  $\tilde{\mathcal{C}}$  be arbitrary nonempty cones. Then the following hold:*

- (i)  $\mathcal{C}^*$  is a closed convex cone.
- (ii)  $\mathcal{C} \subseteq \tilde{\mathcal{C}}$  implies  $\tilde{\mathcal{C}}^* \subseteq \mathcal{C}^*$ .
- (iii)  $\mathcal{C} \subseteq \mathcal{C}^{**}$ .
- (iv) If  $\mathcal{C}$  is convex, then  $\mathcal{C}^{**} = \overline{\mathcal{C}}$ .
- (v)  $\mathcal{C}^{**} = \overline{\text{conv}(\mathcal{C})}$ .

Now, after first stating the following lemma, we will use the dual cone to deduce an equivalent formulation of GCQ in Corollary 3.8. The statements of the following Lemma and Corollary should both be known, but we were not able to find an explicit reference. Therefore, we include their short proofs here.

**Lemma 3.7** *The following equality holds:*

$$\mathcal{T}(z^*)^* = \mathcal{T}^G(z^*)^*. \quad (18)$$

**Proof.** Consider  $\mathcal{T}^G(z^*) = \overline{\text{conv}(\mathcal{T}(z^*))} = \mathcal{T}(z^*)^{**}$ , where we used Lemma 3.6 (v) for the second equality. We dualize this to start off the following string of equations. Roman numerals indicate which point of Lemma 3.6 was used for that particular equality:

$$\mathcal{T}^G(z^*)^* = \mathcal{T}(z^*)^{***} \stackrel{(i),(iv)}{=} \overline{\mathcal{T}(z^*)^*} \stackrel{(i)}{=} \mathcal{T}(z^*)^*.$$

This completes the proof. □

**Corollary 3.8** *Let  $z^*$  be a feasible point of the MPEC (1). Then GCQ holds in  $z^*$  if and only if*

$$\mathcal{T}(z^*)^* = \mathcal{T}^{lin}(z^*)^* \quad (19)$$

*holds.*

**Proof.** Dualizing (9) and using Lemma 3.7 yields  $\mathcal{T}(z^*)^* = \mathcal{T}^G(z^*)^* = \mathcal{T}^{lin}(z^*)^*$ , i.e. GCQ in the form (9) implies (19). To prove the converse implication, the following string of equalities again uses roman numerals to indicate which point of Lemma 3.6 is used. In addition, the equality marked with (\*) is acquired by dualizing (19).

$$\mathcal{T}^{lin}(z^*) \stackrel{(iv)}{=} \mathcal{T}^{lin}(z^*)^{**} \stackrel{(*)}{=} \mathcal{T}(z^*)^{**} \stackrel{(v)}{=} \overline{\text{conv}(\mathcal{T}(z^*))} = \mathcal{T}^G(z^*).$$

For the first equality note that  $\mathcal{T}^{lin}(z^*)$  is polyhedral and as such closed and convex. □

Sometimes, the formulation of the GCQ given in Corollary 3.8 is used to define GCQ (see, e.g. [20, 1]). Also, the formulation (19) of GCQ has been used in [7] in order to characterize the existence of KKT points. As we will see, this formulation of GCQ also facilitates some of the proofs we present in this paper.

## 4 Sufficient Conditions for Guignard CQ

Before we delve into the problem of investigating when GCQ might hold, let us consider necessary conditions for optimality under GCQ. The following theorem is a result from standard nonlinear programming (see [8, 16]).

**Theorem 4.1** *Let  $z^*$  be a  $B$ -stationary point of the MPEC (1). If GCQ holds in  $z^*$ , then there exists a Lagrange multiplier  $\lambda^*$  such that  $(z^*, \lambda^*)$  is a KKT point of (1).*

Now consider a KKT point  $(z, \hat{\lambda})$  of the MPEC (1) with  $\hat{\lambda} = (\hat{\lambda}^g, \hat{\lambda}^h, \hat{\lambda}^G, \hat{\lambda}^H, \hat{\lambda}^\theta)$ . Since  $z$  is, in particular, feasible, we shall neglect conditions which pertain only to feasibility. Setting  $\theta(z) := G(z)^T H(z)$  yields the following representation of the essential conditions for a KKT point:

$$\begin{aligned}
0 &= \nabla f(z) + \sum_{i=1}^m \hat{\lambda}_i^g \nabla g_i(z) + \sum_{i=1}^p \hat{\lambda}_i^h \nabla h_i(z) - \sum_{i=1}^l [\hat{\lambda}_i^G \nabla G_i(z) + \hat{\lambda}_i^H \nabla H_i(z)] + \hat{\lambda}^\theta \nabla \theta(z), \\
G(z) &\geq 0, \quad \hat{\lambda}^G \geq 0, \quad (\hat{\lambda}^G)^T G(z) = 0, \\
H(z) &\geq 0, \quad \hat{\lambda}^H \geq 0, \quad (\hat{\lambda}^H)^T H(z) = 0, \\
g(z) &\leq 0, \quad \hat{\lambda}^g \geq 0, \quad (\hat{\lambda}^g)^T g(z) = 0.
\end{aligned} \tag{20}$$

Keeping in mind that

$$\nabla \theta(z) = \sum_{i=1}^l [G_i(z) \nabla H_i(z) + H_i(z) \nabla G_i(z)],$$

we order the sums in the first line of (20) by gradient. Setting  $\lambda^g := \hat{\lambda}^g$ ,  $\lambda^h := \hat{\lambda}^h$ ,  $\lambda_\alpha^G := \hat{\lambda}_\alpha^G - \hat{\lambda}^\theta H_\alpha(z)$ ,  $\lambda_{\gamma \cup \beta}^G := \hat{\lambda}_{\gamma \cup \beta}^G$ ,  $\lambda_{\alpha \cup \beta}^H := \hat{\lambda}_{\alpha \cup \beta}^H$ , and  $\lambda_\gamma^H := \hat{\lambda}_\gamma^H - \hat{\lambda}^\theta G_\gamma(z)$  then yields the following representation of (20):

$$\begin{aligned}
0 &= \nabla f(z) + \sum_{i=1}^m \lambda_i^g \nabla g_i(z) + \sum_{i=1}^p \lambda_i^h \nabla h_i(z) - \sum_{i=1}^l [\lambda_i^G \nabla G_i(z) + \lambda_i^H \nabla H_i(z)], \\
\lambda_\alpha^G &\text{ free}, \quad \lambda_\beta^G \geq 0, \quad \lambda_\gamma^G = 0, \\
\lambda_\gamma^H &\text{ free}, \quad \lambda_\beta^H \geq 0, \quad \lambda_\alpha^H = 0, \\
g(z) &\leq 0, \quad \lambda^g \geq 0, \quad (\lambda^g)^T g(z) = 0.
\end{aligned} \tag{21}$$

Note that we exploit the complementarity terms in (20) and the nature of the sets  $\alpha$  and  $\gamma$  to get  $\lambda_\gamma^G = 0$  and  $\lambda_\alpha^H = 0$ .

Based on the representation (21), a point  $(z, \lambda)$  with  $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H)$  is called *strongly stationary* [18] or *primal-dual stationary* [15] if  $z$  is feasible for the MPEC (1) and  $(z, \lambda)$  satisfies the conditions (21).

In the context of MPECs, it is common to refer to the concept of strong stationarity and the representation (21) that goes with it, rather than to a KKT point.

The above arguments are collected in the following theorem.

**Theorem 4.2** *Let  $z^*$  be a B-stationary point of the MPEC (1). If GCQ holds in  $z^*$ , then there exists a Lagrange multiplier  $\lambda^*$  such that  $(z^*, \lambda^*)$  is strongly stationary (see (21)).*

We will now investigate the Guignard CQ's position in relation to other CQs which have been examined in the context of MPECs. To this end, let us first introduce a CQ which has been discussed extensively in the past (see, e.g., [18, 5]).

**Definition 4.3** *Let  $z^*$  be feasible for the MPEC (1). Then the MPEC-LICQ is said to hold if the gradient vectors*

$$\begin{aligned} \nabla g_i(z^*) & \quad \forall i \in \mathcal{I}_g := \{i \mid g_i(z^*) = 0\}, \\ \nabla h_i(z^*) & \quad \forall i = 1, \dots, p, \\ \nabla G_i(z^*) & \quad \forall i \in \alpha \cup \beta, \\ \nabla H_i(z^*) & \quad \forall i \in \gamma \cup \beta \end{aligned} \tag{22}$$

*are linearly independent.*

We will now use Corollary 3.8 to show that MPEC-LICQ implies GCQ. Before we do so, however, we state the following lemma, which will facilitate the proof of Theorem 4.5.

**Lemma 4.4** *Let the cones*

$$\begin{aligned} \mathcal{K}_1 := \{d \in \mathbb{R}^n \mid & a_i^T d \geq 0, \quad \forall i = 1, \dots, k, \\ & b_j^T d = 0, \quad \forall j = 1, \dots, l\} \end{aligned} \tag{23}$$

*and*

$$\begin{aligned} \mathcal{K}_2 = \{v \in \mathbb{R}^n \mid v = & \sum_{i=1}^k \alpha_i a_i + \sum_{j=1}^l \beta_j b_j \\ & \alpha_i \geq 0, \quad \forall i = 1, \dots, k\}, \end{aligned} \tag{24}$$

*be given. Then  $\mathcal{K}_1 = \mathcal{K}_2^*$  and  $\mathcal{K}_1^* = \mathcal{K}_2$ .*

**Proof.** See Theorem 3.2.2 in [1]. □

We now use Lemma 4.4 to prove the following theorem.

**Theorem 4.5** *If a feasible point  $z^*$  of the MPEC (1) satisfies MPEC-LICQ, it also satisfies GCQ.*

**Proof.** We will show that MPEC-LICQ implies the equivalent definition of GCQ introduced in Corollary 3.8. It is well-known that  $\mathcal{T}(z^*) \subseteq \mathcal{T}^{lin}(z^*)$ . Dualizing this yields  $\mathcal{T}^{lin}(z^*)^* \subseteq \mathcal{T}(z^*)^*$ . Hence it suffices to show that

$$\mathcal{T}(z^*)^* \subseteq \mathcal{T}^{lin}(z^*)^* \quad (25)$$

holds. As mentioned in the proof of Proposition 2.1 (see (8)), it holds that

$$\mathcal{T}(z^*) = \bigcup_{(\beta_1, \beta_2) \in \mathcal{P}(\beta)} \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}(z^*).$$

Dualizing this yields

$$\mathcal{T}(z^*)^* = \bigcap_{(\beta_1, \beta_2) \in \mathcal{P}(\beta)} \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}(z^*)^* \quad (26)$$

(see [1, Theorem 3.1.9]).

Since MPEC-LICQ holds for the MPEC (1), standard LICQ and hence ACQ holds for each  $\text{NLP}_*(\beta_1, \beta_2)$ , i.e. we have

$$\mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}(z^*) = \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{lin}(z^*) \quad (27)$$

for every  $(\beta_1, \beta_2) \in \mathcal{P}(\beta)$ . Thus, we can apply Lemma 4.4 to the representation (7) of  $\mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{lin}(z^*)$ , yielding the dual of  $\mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}(z^*)$  as follows:

$$\begin{aligned} \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}(z^*)^* &= \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}^{lin}(z^*)^* \\ &= \{v \in \mathbb{R}^n \mid v = - \sum_{i \in \mathcal{I}_g} u_i^g \nabla g_i(z^*) - \sum_{i=1}^p u_i^h \nabla h_i(z^*) \\ &\quad + \sum_{i \in \alpha \cup \beta} u_i^G \nabla G_i(z^*) + \sum_{i \in \gamma \cup \beta} u_i^H \nabla H_i(z^*), \\ &\quad u_{\mathcal{I}_g}^g \geq 0, \quad u_{\beta_2}^G \geq 0, \quad u_{\beta_1}^H \geq 0\}. \end{aligned} \quad (28)$$

Taking  $v \in \mathcal{T}(z^*)^*$  arbitrarily, (26) yields that

$$v \in \mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}(z^*)^* \quad \text{and} \quad v \in \mathcal{T}_{\text{NLP}_*(\beta_2, \beta_1)}(z^*)^*$$

for an arbitrary partition  $(\beta_1, \beta_2) \in \mathcal{P}(\beta)$  and its ‘‘complement’’  $(\beta_2, \beta_1) \in \mathcal{P}(\beta)$ .

Since all gradient vectors in (28) are linearly independent (MPEC-LICQ holds),  $u_{\mathcal{I}_g}^g$ ,  $u^h$ ,  $u_{\alpha \cup \beta}^G$ , and  $u_{\gamma \cup \beta}^H$  are uniquely defined. Hence it follows from the fact that  $v$  is in both  $\mathcal{T}_{\text{NLP}_*(\beta_1, \beta_2)}(z^*)^*$  and  $\mathcal{T}_{\text{NLP}_*(\beta_2, \beta_1)}(z^*)^*$  that  $u_{\beta}^G \geq 0$  and  $u_{\beta}^H \geq 0$ . Therefore,

$$\begin{aligned} v &\in \{v \in \mathbb{R}^n \mid v = - \sum_{i \in \mathcal{I}_g} u_i^g \nabla g_i(z^*) - \sum_{i=1}^p u_i^h \nabla h_i(z^*) \\ &\quad + \sum_{i \in \alpha \cup \beta} u_i^G \nabla G_i(z^*) + \sum_{i \in \gamma \cup \beta} u_i^H \nabla H_i(z^*), \\ &\quad u_{\mathcal{I}_g}^g \geq 0, \quad u_{\beta}^G \geq 0, \quad u_{\beta}^H \geq 0\} \\ &= \mathcal{T}^{lin}(z^*)^*, \end{aligned}$$

which proves (25). (The above representation of  $\mathcal{T}^{lin}(z^*)^*$  can be gleaned by applying Lemma 4.4 to the representation (5) of  $\mathcal{T}^{lin}(z^*)$ .)  $\square$

We have now shown MPEC-LICQ to imply GCQ, and also strong stationarity to be a necessary optimality condition under GCQ (Theorems 4.5 and 4.2 respectively). This is (except for uniqueness of the Lagrange multiplier, which follows trivially from MPEC-LICQ) the statement of [5, Theorem 3.4].

**Remark.** It is well known (see, e.g., [18, 5]) that strong stationarity is a necessary first order condition under MPEC-SMFCQ (the MPEC variant of the strict Mangasarian-Fromovitz constraint qualification, see, e.g., [10] for the standard SMFCQ).

In the face of Theorem 4.2 and the fact that MPEC-SMFCQ implies uniqueness of the Lagrange multiplier, it stands to reason that MPEC-SMFCQ should imply GCQ. Note that MPEC-SMFCQ is weaker than MPEC-LICQ, so this question is of some interest.

This can in fact be deduced by the following string of reasoning: Since strong stationarity is a necessary first order condition under MPEC-SMFCQ, also a KKT point (20) is necessary under MPEC-SMFCQ (the reasoning following Theorem 4.1 shows that a strongly stationary point (21) is equivalent to a KKT point (20) of the MPEC (1)). A result by Gould and Tolle (see the Theorem in Section 3 of [7] or [1, Theorem 6.3.2]) states that the existence of such a KKT point (for an arbitrary objective function  $f$ , but MPEC-SMFCQ is stated independent of the objective function) implies GCQ.

We note, however, that we were not able to find a more direct and elementary proof of the fact that MPEC-SMFCQ implies GCQ.

Borrowing heavily from [15, Theorem 1], we will now show GCQ to hold under relatively weak, though not very tangible, assumptions. In the following we will introduce these assumptions, after which we will be able to prove our variant of [15, Theorem 1].

In [4] we introduced the *MPEC-linearized tangent cone*,

$$\begin{aligned}
\mathcal{T}_{\text{MPEC}}^{lin}(z^*) := \{d \in \mathbb{R}^n \mid & \nabla g_i(z^*)^T d \leq 0 & \forall i \in \mathcal{I}_g, \\
& \nabla h_i(z^*)^T d = 0 & \forall i = 1, \dots, p, \\
& \nabla G_i(z^*)^T d = 0 & \forall i \in \alpha, \\
& \nabla H_i(z^*)^T d = 0 & \forall i \in \gamma, \\
& \nabla G_i(z^*)^T d \geq 0 & \forall i \in \beta, \\
& \nabla H_i(z^*)^T d \geq 0 & \forall i \in \beta, \\
& (\nabla G_i(z^*)^T d) \cdot (\nabla H_i(z^*)^T d) = 0, & \forall i \in \beta\}.
\end{aligned} \tag{29}$$

Note that  $\mathcal{T}_{\text{MPEC}}^{lin}(z^*)$  is different from the standard linearized tangent cone  $\mathcal{T}^{lin}(z^*)$  and that we always have  $\mathcal{T}(z^*) \subseteq \mathcal{T}_{\text{MPEC}}^{lin}(z^*)$  (see Lemma 3.2 in [4]). Hence, the definition of the *MPEC-Abadie constraint qualification*

$$\mathcal{T}(z^*) = \mathcal{T}_{\text{MPEC}}^{lin}(z^*) \tag{30}$$

(referred to as MPEC-ACQ in the following) makes sense. In [4] we were able to prove A-stationarity (weaker than strong stationarity) to be a necessary optimality condition under MPEC-ACQ. It is important to note that  $\mathcal{T}_{\text{MPEC}}^{\text{lin}}(z^*)$  can be expressed as follows:

$$\mathcal{T}_{\text{MPEC}}^{\text{lin}}(z^*) = \bigcup_{(\beta_1, \beta_2) \in \mathcal{P}(\beta)} \mathcal{T}_{\text{NLP}^*(\beta_1, \beta_2)}^{\text{lin}}(z^*). \quad (31)$$

See the treatise [4] for a more detailed discussion of MPEC-ACQ.

For the second assumption of Theorem 4.7, we must first introduce the concept of nonsingularity, as used in [15, 21].

**Definition 4.6** *Given the linear system*

$$Ax \leq b, \quad Cx = d, \quad (32)$$

*an inequality  $a_i x \leq b_i$  is said to be nonsingular if there exists a feasible solution of the system (32) which satisfies this inequality strictly. Here  $a_i$  denotes the  $i$ -th row of the matrix  $A$ .*

We will now apply nonsingularity to the linearized tangent cone  $\mathcal{T}^{\text{lin}}(z^*)$  in the point  $z^*$  (see (5)). To this end we introduce two new sets: Let  $\beta^G$  denote the subset of  $\beta$  consisting of all indices  $i \in \beta$  such that the inequality  $\nabla G_i(z^*)^T d \geq 0$  is nonsingular in the system defining  $\mathcal{T}^{\text{lin}}(z^*)$ . Similarly, we denote by  $\beta^H$  the nonsingular set pertaining to the inequalities  $\nabla H_i(z^*)^T d \geq 0$ . Note that  $\beta^G$  and  $\beta^H$  depend on  $z^*$ .

Using the sets  $\beta^G$  and  $\beta^H$  renders the following representation of  $\mathcal{T}^{\text{lin}}(z^*)$  (cf. (5)):

$$\begin{aligned} \mathcal{T}^{\text{lin}}(z^*) = \{d \in \mathbb{R}^n \mid & \nabla g_i(z^*)^T d \leq 0 & \forall i \in \mathcal{I}_g, \\ & \nabla h_i(z^*)^T d = 0 & \forall i = 1, \dots, p, \\ & \nabla G_i(z^*)^T d = 0 & \forall i \in \alpha \cup \beta \setminus \beta^G, \\ & \nabla H_i(z^*)^T d = 0 & \forall i \in \gamma \cup \beta \setminus \beta^H, \\ & \nabla G_i(z^*)^T d \geq 0 & \forall i \in \beta^G, \\ & \nabla H_i(z^*)^T d \geq 0 & \forall i \in \beta^H \}. \end{aligned} \quad (33)$$

We will now also use the sets  $\beta^G$  and  $\beta^H$  to define the following assumption (A2). Note that (A2) is equivalent to [15, (A2)] by Lemma 1 of the same reference.

**(A2)** Given the feasible point  $z^*$ , there exists a partition  $(\beta_1^{GH}, \beta_2^{GH}) \in \mathcal{P}(\beta^G \cap \beta^H)$  such that

(A2a) for each  $i_0 \in \beta_1^{GH}$  there exists a vector  $d$  such that

$$\begin{aligned} \nabla G_{i_0}(z^*)^T d &> 0, \\ \nabla G_i(z^*)^T d &= 0 & \forall i \in \alpha \cup \beta \setminus \{i_0\}, \\ \nabla H_i(z^*)^T d &= 0 & \forall i \in \gamma \cup \beta, \\ \nabla g_i(z^*)^T d &= 0 & \forall i \in \mathcal{I}_g, \\ \nabla h_i(z^*)^T d &= 0 & \forall i = 1, \dots, p; \end{aligned} \quad (34)$$

(A2b) for each  $i_0 \in \beta_2^{GH}$  there exists a vector  $d$  such that

$$\begin{aligned}
\nabla H_{i_0}(z^*)^T d &> 0, \\
\nabla H_i(z^*)^T d &= 0 & \forall i \in \gamma \cup \beta \setminus \{i_0\}, \\
\nabla G_i(z^*)^T d &= 0 & \forall i \in \alpha \cup \beta, \\
\nabla g_i(z^*)^T d &= 0 & \forall i \in \mathcal{I}_g, \\
\nabla h_i(z^*)^T d &= 0 & \forall i = 1, \dots, p.
\end{aligned} \tag{35}$$

We have finally collected enough information to state and prove the following theorem.

**Theorem 4.7** *If a feasible point  $z^*$  of the MPEC (1) satisfies both MPEC-ACQ and assumption (A2), it also satisfies GCQ.*

**Proof.** As in the proof to Theorem 4.5, it suffices to show that the inclusion

$$\mathcal{T}(z^*)^* \subseteq \mathcal{T}^{lin}(z^*)^* \tag{36}$$

holds (see (25)). Since MPEC-ACQ holds, we have

$$\mathcal{T}(z^*) = \mathcal{T}_{\text{MPEC}}^{lin}(z^*) = \bigcup_{(\beta_1, \beta_2) \in \mathcal{P}(\beta)} \mathcal{T}_{\text{NLP}^*(\beta_1, \beta_2)}^{lin}(z^*), \tag{37}$$

see (30) and (31). Similar to (26), we dualize (37) using [1, Theorem 3.1.9], yielding

$$\mathcal{T}(z^*)^* = \mathcal{T}_{\text{MPEC}}^{lin}(z^*)^* = \bigcap_{(\beta_1, \beta_2) \in \mathcal{P}(\beta)} \mathcal{T}_{\text{NLP}^*(\beta_1, \beta_2)}^{lin}(z^*)^*. \tag{38}$$

Now to prove (36), we take an arbitrary  $v \in \mathcal{T}(z^*)^*$ . By virtue of (38), we have

$$v \in \mathcal{T}_{\text{NLP}^*(\beta_1, \beta_2)}^{lin}(z^*)^* \quad \forall (\beta_1, \beta_2) \in \mathcal{P}(\beta). \tag{39}$$

Consider the specific partition of  $\beta$  given by

$$\hat{\beta}_1 := \beta^H \setminus \beta_2^{GH}, \quad \hat{\beta}_2 := \beta \setminus \hat{\beta}_1, \tag{40}$$

Here  $(\beta_1^{GH}, \beta_2^{GH}) \in \mathcal{P}(\beta^G \cap \beta^H)$  is a partition of  $\beta^G \cap \beta^H$  that satisfies assumption (A2). Note that  $\beta^G \setminus \beta_1^{GH} \subseteq \hat{\beta}_2$ .

Since  $v \in \mathcal{T}_{\text{NLP}^*(\hat{\beta}_1, \hat{\beta}_2)}^{lin}(z^*)^*$ , we can apply Lemma 4.4, yielding the existence of a vector  $u = (u^g, u^h, u^G, u^H)$  with

$$\begin{aligned}
u_i^g &\geq 0 & \forall i \in \mathcal{I}_g, \\
u_i^G &\geq 0 & \forall i \in \hat{\beta}_2, \\
u_i^H &\geq 0 & \forall i \in \hat{\beta}_1
\end{aligned}$$

such that

$$v = - \sum_{i \in \mathcal{I}_g} u_i^g \nabla g_i(z^*) - \sum_{i=1}^p u_i^h \nabla h_i(z^*) + \sum_{i \in \alpha \cup \beta} u_i^G \nabla G_i(z^*) + \sum_{i \in \gamma \cup \beta} u_i^H \nabla H_i(z^*). \quad (41)$$

The choice of the sets  $\hat{\beta}_1$  and  $\hat{\beta}_2$  guarantee, in particular, that

$$u_i^G \geq 0 \quad \forall i \in \beta^G \setminus \beta_1^{GH} \quad \text{and} \quad u_i^H \geq 0 \quad \forall i \in \beta^H \setminus \beta_2^{GH}.$$

Due to the representation (33) of  $\mathcal{T}^{lin}(z^*)$  and using Lemma 4.4 to represent  $\mathcal{T}^{lin}(z^*)^*$  it therefore suffices to show that

$$u_i^G \geq 0 \quad \forall i \in \beta_1^{GH} \quad \text{and} \quad u_i^H \geq 0 \quad \forall i \in \beta_2^{GH}. \quad (42)$$

Since the proof is similar for  $u_i^G$  and  $u_i^H$ , we will only demonstrate the former. Let therefore  $u_{i_0}^G < 0$  for some index  $i_0 \in \beta_1^{GH}$ . Corresponding to this index, let  $d$  be the vector satisfying (34) in assumption (A2a). Multiplying the vector  $v$  from (41) by this  $d$  yields

$$v^T d = \underbrace{u_{i_0}^G}_{<0} \underbrace{\nabla G_{i_0}(z^*)^T d}_{>0} < 0. \quad (43)$$

By comparing representation (34) of  $d$  with the representation (7) of  $\mathcal{T}_{\text{NLP}^*(\beta_1, \beta_2)}^{lin}(z^*)$ , we see that  $d \in \mathcal{T}_{\text{NLP}^*(\beta_1, \beta_2)}^{lin}(z^*)$  for all partitions  $(\beta_1, \beta_2) \in \mathcal{P}(\beta)$  such that  $i_0 \in \beta_2$ . For any such partition  $(\beta_1, \beta_2)$ , it holds that  $v \in \mathcal{T}_{\text{NLP}^*(\beta_1, \beta_2)}^{lin}(z^*)^*$  by (39), and hence we have  $v^T d \geq 0$  which is a contradiction to (43), yielding  $u_i^G \geq 0$ . This completes the proof.  $\square$

The proof of Theorem 4.7 is closely related to the proof of Theorem 1 in Pang and Fukushima's paper [15]. In fact, the only difference in the statement of our theorem and theirs is the relaxation of one of the assumptions: where we show MPEC-ACQ and (A2) to suffice, they require (A1) and (A2). (Note that (A1) obviously implies MPEC-ACQ.)

**Remark.** It is of some interest that the proofs of Theorems 4.5 and 4.7 differ substantially. The technique of the proof of Theorem 4.5 can, however, be applied to Theorem 4.7, if assumption (A2) is replaced by the *partial MPEC-LICQ* which is said to hold if for every vector  $u := (u^g, u^h, u^G, u^H)$  satisfying

$$0 = \sum_{i \in \mathcal{I}_g} u_i^g \nabla g_i(z^*) + \sum_{i=1}^p u_i^h \nabla h_i(z^*) + \sum_{i \in \alpha \cup \beta} u_i^G \nabla G_i(z^*) + \sum_{i \in \gamma \cup \beta} u_i^H \nabla H_i(z^*), \quad (44)$$

it is implied that  $u_\beta^G = u_\beta^H = 0$  (see, e.g., [22]). Hence, using the technique of the proof of Theorem 4.5, it can be shown that if a feasible point  $z^*$  satisfies both MPEC-ACQ and partial MPEC-LICQ, then it also satisfies GCQ.

Note that in general, partial MPEC-LICQ is a stronger assumption than (A2), since according to Lemma 1 in [15], (A2) is equivalent to stating that (44) implies that  $u_{\beta_1^{GH}}^G = u_{\beta_2^{GH}}^H = 0$  (note that  $\beta_1^{GH}, \beta_2^{GH} \in \beta$ ).



## 5 Conclusion

The goal of this paper was to clarify the position of Guignard CQ in the context of MPECs and put it in relation to other CQs examined by others [15, 18].

In this context we recapitulated why other constraint qualifications known from standard nonlinear programming fail to hold for MPECs. Even Adabie CQ, which is one of the weaker CQs, was shown to hold for MPECs only under very restrictive circumstances.

**Acknowledgements.** We would like to thank Danny Ralph and Jong-Shi Pang for first suggesting to investigate the Guignard constraint qualification. Again, we are grateful to Marco López for supplying us with Example 3.5. Involved in the creation of Figure 1 for this example was Manfred Dobrowolski, whom, with this, we also wish to thank.

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