ON THE ABADIE AND GUIGNARD CONSTRAINT QUALIFICATIONS FOR MATHEMATICAL PROGRAMS WITH VANISHING CONSTRAINTS

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Abstract. We consider a special class of optimization problems that we call a Mathematical Program with Vanishing Constraints. It has a number of important applications in structural and topology optimization, but typically does not satisfy standard constraint qualifications like the linear independence and the Mangasarian-Fromovitz constraint qualification. We therefore investigate the Abadie and Guignard constraint qualifications in more detail. In particular, it follows from our results that also the Abadie constraint qualification is typically not satisfied, whereas the Guignard constraint qualification holds under fairly mild assumptions for our particular class of optimization problems.

Key Words: Mathematical programs with vanishing constraints, Mathematical programs with equilibrium constraints, Abadie constraint qualification, Guignard constraint qualification

1 Introduction

We consider a constrained optimization problem of the form

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \le 0 \quad \forall i = 1, \ldots, m, \\ & h_j(x) = 0 \quad \forall j = 1, \ldots, p, \\ & H_i(x) \ge 0 \quad \forall i = 1, \ldots, l, \\ & G_i(x) H_i(x) \le 0 \quad \forall i = 1, \ldots, l \end{array} \tag{1}$$

that we call a *Mathematical Program with Vanishing Constraints*, or MPVC for short, where all functions $f, g_i, h_j, H_i, G_i : \mathbb{R}^n \to \mathbb{R}$ are assumed to be continuously differentiable. This special class of optimization problems was recently introduced in [2] and shown to act as a unified framework for several applications in structural and topology optimization.

The MPVC is closely related to the more commonly known *Mathematical Program with Equilibrium Constraints*, MPEC for short, that has the form

$$\begin{array}{ll} \min & \tilde{f}(z) \\ \text{s.t.} & \tilde{g}_i(z) \le 0 \quad \forall i = 1, \ldots, \tilde{m}, \\ & \tilde{h}_j(z) = 0 \quad \forall j = 1, \ldots, \tilde{p}, \\ & \tilde{G}_i(z) \ge 0, \ \tilde{H}_i(z) \ge 0, \ \tilde{G}_i(z) \tilde{H}_i(z) = 0 \quad \forall i = 1, \ldots, \tilde{l} \end{array}$$

for certain functions $\tilde{f}, \tilde{g}_i, \tilde{h}_j, \tilde{G}_i, \tilde{H}_i : \mathbb{R}^{\tilde{n}} \to \mathbb{R}$, see, for example, the two books [10, 13] for more details. In fact, it was noted in [2] that an MPVC can always be reformulated as an MPEC, however, this reformulation has some disadvantages since it increases the dimension of the problem and, more importantly, since it involves a nonuniqueness of the solutions. Moreover, viewing an MPVC as an MPEC does not take into account the special structure of an MPVC. This, however, is highly recommended since some preliminary results in [2] indicate that an MPVC is somewhat simpler than an MPEC.

Nevertheless, it was already noted in [2] that also the MPVC is a difficult optimization problem. For example, both the LICQ (linear independence constraint qualification) and the MFCQ (Mangasarian-Fromovitz constraint qualification) were shown to be violated under fairly mild conditions. (Note that, on the other hand, both LICQ and MFCQ are *always* violated for an MPEC, see [4].) A natural question that we try to answer in this paper therefore is: What about weaker constraint qualifications? In particular, we discuss in more detail the ACQ (Abadie constraint qualification) and the GCQ (Guignard constraint qualification) introduced in [1] and [9], respectively.

Apart from the LICQ and MFCQ, the ACQ is probably the most prominent constraint qualification that is used for standard optimization problems and can be found in many textbooks like, for example, [12]. The GCQ is much less known and can hardly be found in any textbook, however, it was noted in [8] that it is the weakest constraint qualification which guarantees that, at a local minimum of an optimization problem, there exist Lagrange multipliers such that the Karush-Kuhn-Tucker conditions are first-order optimality conditions.

The organization of this paper is as follows: We begin with some preliminary results in Section 2. Section 3 then states some necessary conditions for the ACQ to be satisfied. These conditions indicate that ACQ is unlikely to hold at a local minimum (or any feasible point) of the MPVC. We therefore take a closer look at the GCQ in the following two sections and present some sufficient conditions for the GCQ to be satisfied. First, we give a relatively simple LICQ-type condition in Section 4, and then we refine our analysis and state a weaker condition in Section 5. These sufficient conditions indicate that GCQ is satisfied in many important situations. The analysis carried out in Sections 3–5 is motivated by some corresponding results for MPECs given in [14, 6]. We close with some final remarks in Section 6.

The notation used in this paper is standard. The only thing we would like to mention here is that $\mathcal{P}(J)$ denotes the set of all partitions of a given set J, i.e., the set of all pairs (J_1, J_2) such that $J_1 \cap J_2 = \emptyset$ and $J_1 \cup J_2 = J$.

2 Preliminaries

In this section we recall some basic definitions from optimization, introduce several index sets and state some preliminary results that will play an important role in our subsequent analysis.

We begin with a formal definition of the dual cone which is the negative of the more commonly known polar cone, see [3, 16] for a further discussion.

Definition 2.1 *Let $\mathcal{C} \subseteq \mathbb{R}^n$ be an arbitrary cone. Then $\mathcal{C}^* := \{v \in \mathbb{R}^n \,|\, v^T d \geq 0 \ \ \forall d \in \mathcal{C}\}$ denotes the* dual cone *of $\mathcal{C}$.*

Next consider a general optimization problem of the form

$$\begin{array}{ll} \min & \tilde{f}(x) \\ \text{s.t.} & \tilde{g}_i(x) \leq 0 \quad \forall i = 1, \ldots, \tilde{m}, \\ & \tilde{h}_j(x) = 0 \quad \forall j = 1, \ldots, \tilde{p}, \end{array} \tag{2}$$

where all functions $\tilde{f}, \tilde{g}_i, \tilde{h}_j : \mathbb{R}^{\tilde{n}} \to \mathbb{R}$ are assumed to be continuously differentiable. Let $\tilde{X}$ denote the feasible set of this optimization problem. Then the *tangent cone* at a feasible point $\tilde{x} \in \tilde{X}$ is defined by

$$\mathcal{T}(\tilde{x}) := \big\{d \in \mathbb{R}^{\tilde{n}} \;\big|\; \exists \{x^k\} \subseteq \tilde{X}, t_k \downarrow 0 : x^k \to \tilde{x} \text{ and } \frac{x^k - \tilde{x}}{t_k} \to d\big\}.$$

Furthermore, the *linearized cone* at $\tilde{x} \in \tilde{X}$ is defined by

$$\begin{aligned} \mathcal{L}(\tilde{x}) = \big\{d \in \mathbb{R}^{\tilde{n}} \;|\; & \nabla \tilde{g}_i(\tilde{x})^T d \leq 0 \quad (i : \tilde{g}_i(\tilde{x}) = 0), \\ & \nabla \tilde{h}_j(\tilde{x})^T d = 0 \quad (j = 1, \ldots, \tilde{p})\big\}. \end{aligned}$$

Then the following definitions are standard in optimization, see, e.g., [3, 15].

Definition 2.2 *Let $\tilde{x} \in \tilde{X}$ be a feasible point of the program* (2)*. Then*

(a) the Abadie constraint qualification *(ACQ for short) holds at $\tilde{x}$ if $\mathcal{L}(\tilde{x}) = \mathcal{T}(\tilde{x})$.*

(b) the Guignard constraint qualification *(GCQ for short) holds at $\tilde{x}$ if $\mathcal{L}(\tilde{x})^* = \mathcal{T}(\tilde{x})^*$.*

ACQ obviously implies GCQ, whereas the converse is not true in general, see [15] for a counterexample.

We now come back to our MPVC from (1). In order to state a representation of the linearized cone for this specially structured optimization problem, we need to introduce a number of index sets. To this end, let X denote the feasible set of (1), and let $x^* \in X$ be an arbitrary feasible point. Then we define the index sets

$$\begin{array}{rcl} I_g & := & \big\{ i \,\big|\, g_i(x^*) = 0 \big\}, \\ I_+ & := & \big\{ i \,\big|\, H_i(x^*) > 0 \big\}, \\ I_0 & := & \big\{ i \,\big|\, H_i(x^*) = 0 \big\}. \end{array} \tag{3}$$

Furthermore, we divide the index set I_+ into the following subsets:

$$\begin{array}{rcl} I_{+0} & := & \big\{ i \,\big|\, H_i(x^*) > 0, G_i(x^*) = 0 \big\}, \\ I_{+-} & := & \big\{ i \,\big|\, H_i(x^*) > 0, G_i(x^*) < 0 \big\}. \end{array} \tag{4}$$

Similarly, we partition the set I_0 in the following way:

$$\begin{array}{rcl} I_{0+} & := & \big\{ i \,\big|\, H_i(x^*) = 0, G_i(x^*) > 0 \big\}, \\ I_{00} & := & \big\{ i \,\big|\, H_i(x^*) = 0, G_i(x^*) = 0 \big\}, \\ I_{0-} & := & \big\{ i \,\big|\, H_i(x^*) = 0, G_i(x^*) < 0 \big\}. \end{array} \tag{5}$$

Note that the first subscript indicates the sign of $H_i(x^*)$, whereas the second subscript stands for the sign of $G_i(x^*)$. Using these index sets, we can state the following representation of the linearized cone at a feasible point of our MPVC. Its elementary proof can be found in [2, Lemma 4].

Lemma 2.3 *Let $x^* \in X$ be a feasible point for* (1)*. Then the linearized cone at x^* is given by*

$$\begin{array}{rll} \mathcal{L}(x^*) = \big\{ d \in \mathbb{R}^n \mid & \nabla g_i(x^*)^T d \le 0 & (i \in I_g), \\ & \nabla h_j(x^*)^T d = 0 & (j = 1, \ldots, p), \\ & \nabla H_i(x^*)^T d = 0 & (i \in I_{0+}), \\ & \nabla H_i(x^*)^T d \ge 0 & (i \in I_{00} \cup I_{0-}), \\ & \nabla G_i(x^*)^T d \le 0 & (i \in I_{+0}) \big\}. \end{array} \tag{6}$$

Note that the inclusion $\mathcal{T}(x^*) \subseteq \mathcal{L}(x^*)$ always holds and that $\mathcal{T}(x^*)$ is always closed, but not necessarily convex, while $\mathcal{L}(x^*)$ is polyhedral and thus closed and convex.

In order to get a suitable representation of the tangent cone itself, it will be crucial to introduce a certain program derived from our MPVC. To this end, let x^* be feasible for the

program (1), and let $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$ be an arbitrary partition of the index set I_{00} (recall that this index set depends on x^*). Then $NLP_*(\beta_1, \beta_2)$ denotes the nonlinear program

$$\begin{array}{lll} \min & f(x) & \\ \text{s.t.} & g_i(x) \le 0 & \forall i = 1, \ldots, m, \\ & h_j(x) = 0 & \forall j = 1, \ldots, p, \\ & H_i(x) = 0 & \forall i \in I_{0+}, \\ & H_i(x) \ge 0 & \forall i \in I_{0-}, \\ & G_i(x) \le 0 & \forall i \in I_{+0}, \\ & H_i(x) \ge 0 & \forall i \in \beta_1, \\ & G_i(x) \le 0 & \forall i \in \beta_1, \\ & H_i(x) = 0 & \forall i \in \beta_2, \\ & H_i(x) \ge 0 & \forall i \in I_{+}, \\ & G_i(x) \le 0 & \forall i \in I_{+-} \cup I_{0-}. \end{array} \tag{7}$$

The linearized cone of $NLP_*(\beta_1, \beta_2)$ is given by

$$\begin{array}{rll} \mathcal{L}_{NLP_*(\beta_1,\beta_2)}(x^*) = \big\{ d \in \mathbb{R}^n \mid & \nabla g_i(x^*)^T d \le 0 & (i \in I_g), \\ & \nabla h_j(x^*)^T d = 0 & (j = 1, \ldots, p), \\ & \nabla H_i(x^*)^T d = 0 & (i \in I_{0+}), \\ & \nabla H_i(x^*)^T d \ge 0 & (i \in I_{0-}), \\ & \nabla G_i(x^*)^T d \le 0 & (i \in I_{+0}), \\ & \nabla H_i(x^*)^T d \ge 0 & (i \in \beta_1), \\ & \nabla G_i(x^*)^T d \le 0 & (i \in \beta_1), \\ & \nabla H_i(x^*)^T d = 0 & (i \in \beta_2) \big\}. \end{array} \tag{8}$$

A further cone that we will make use of is

$$\begin{array}{rll} \mathcal{L}_{MPVC}(x^*) := \big\{ d \in \mathbb{R}^n \mid & \nabla g_i(x^*)^T d \le 0 & (i \in I_g), \\ & \nabla h_j(x^*)^T d = 0 & (j = 1, \ldots, p), \\ & \nabla H_i(x^*)^T d = 0 & (i \in I_{0+}), \\ & \nabla H_i(x^*)^T d \ge 0 & (i \in I_{00} \cup I_{0-}), \\ & \nabla G_i(x^*)^T d \le 0 & (i \in I_{+0}), \\ & (\nabla H_i(x^*)^T d)(\nabla G_i(x^*)^T d) \le 0 & (i \in I_{00}) \big\}. \end{array} \tag{9}$$

We will call $\mathcal{L}_{MPVC}(x^*)$ the *MPVC-linearized cone* since it takes into account the special structure of the MPVC. Note that it is, in general, a nonconvex cone, and that the only difference between $\mathcal{L}_{MPVC}(x^*)$ and the linearized cone $\mathcal{L}(x^*)$ is that we add a quadratic term in the last line of (9), cf. Lemma 2.3. In particular, we always have the inclusion $\mathcal{L}_{MPVC}(x^*) \subseteq \mathcal{L}(x^*)$.

With the above definitions, we are now in a position to state the following lemma which is the counterpart of corresponding results known from the MPEC literature, see, e.g., [10, 14, 6].

Lemma 2.4 *Let x^* be feasible for* (1)*. Then the following statements hold:*

(a) $\mathcal{T}(x^*) = \bigcup_{(\beta_1,\beta_2)\in\mathcal{P}(I_{00})} \mathcal{T}_{NLP_*(\beta_1,\beta_2)}(x^*)$.

(b) $\mathcal{L}_{MPVC}(x^*) = \bigcup_{(\beta_1,\beta_2)\in\mathcal{P}(I_{00})} \mathcal{L}_{NLP_*(\beta_1,\beta_2)}(x^*)$.

Proof. (a) $'\subseteq'$: Let $d \in \mathcal{T}(x^*)$. Then there exist sequences $\{x^k\} \subseteq X$ and $\{t_k\} \subseteq \mathbb{R}$ with $t_k \downarrow 0$ such that $\frac{x^k-x^*}{t_k} \to d$. Thus, it suffices to show that there exists a partition $(\hat\beta_1, \hat\beta_2) \in \mathcal{P}(I_{00})$ and an infinite set $K \subseteq \mathbb{N}$ such that x^k is feasible for $NLP_*(\hat\beta_1, \hat\beta_2)$ for all $k \in K$. Since x^k is feasible for (1) and all functions are at least continuous, we have $g_i(x^k) \le 0$ $(i = 1, \dots, m)$, $h_j(x^k) = 0$ $(j = 1, \dots, p)$, $H_i(x^k) \ge 0$ $(i \in I_{0-})$, $H_i(x^k) \ge 0$ $(i \in I_+)$ and $G_i(x^k) \le 0$ $(i \in I_{+-} \cup I_{0-})$ for all $k \in \mathbb{N}$ sufficiently large. For $i \in I_{0+}$ we have $G_i(x^k) > 0$ for k sufficiently large, again by continuity. Therefore, we obtain $H_i(x^k) = 0$ for all $i \in I_{0+}$ and all k sufficiently large, as x^k is feasible for (1). Using a similar argument, we also obtain $G_i(x^k) \le 0$ for all $i \in I_{+0}$ for k sufficiently large. Now put

$$\beta_{1,k} := \big\{i \in I_{00} \mid G_i(x^k) \le 0\big\} \quad \text{and} \quad \beta_{2,k} := \big\{i \in I_{00} \mid G_i(x^k) > 0\big\}$$

for all $k \in \mathbb{N}$. Since $\mathcal{P}(I_{00})$ contains only a finite number of partitions, we can find a particular partition $(\hat\beta_1, \hat\beta_2)$ and an infinite set $K \subseteq \mathbb{N}$ such that $(\beta_{1,k}, \beta_{2,k}) = (\hat\beta_1, \hat\beta_2)$ for all $k \in K$. Then $(\hat\beta_1, \hat\beta_2)$ and K have the desired properties.
$'\supseteq'$: For all $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$ one can easily see by the definition of the respective programs that any feasible point of $NLP_*(\beta_1, \beta_2)$ is also feasible for (1). Hence, we obtain $\mathcal{T}(x^*) \supseteq \mathcal{T}_{NLP_*(\beta_1,\beta_2)}(x^*)$ for all $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$, which implies the claimed inclusion.

(b) $'\subseteq'$: Let $d \in \mathcal{L}_{MPVC}(x^*)$. Recalling the definitions of the corresponding linearized cones, we only need to show that there exists a partition $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$ such that $\nabla G_i(x^*)^T d \le 0$ $(i \in \beta_1)$ and $\nabla H_i(x^*)^T d = 0$ $(i \in \beta_2)$ holds, since all other restrictions are trivially satisfied. To this end, put

$$\beta_1 := \big\{i \in I_{00} \mid \nabla G_i(x^*)^T d \le 0\big\}, \quad \beta_2 := \big\{i \in I_{00} \mid \nabla G_i(x^*)^T d > 0\big\}.$$

Since we have $(\nabla H_i(x^*)^T d)(\nabla G_i(x^*)^T d) \le 0$ and $\nabla H_i(x^*)^T d \ge 0$ for all $i \in I_{00}$ by assumption, we can conclude from the above definitions that $\nabla H_i(x^*)^T d = 0$ holds for all $i \in \beta_2$ which proves the first inclusion.
$'\supseteq'$: This inclusion follows immediately from the definitions of the corresponding cones. □

An immediate consequence of the previous result is the following one.

Corollary 2.5 *Let x^* be feasible for* (1)*. Then we have $\mathcal{T}(x^*) \subseteq \mathcal{L}_{MPVC}(x^*)$.*

Proof. Since the tangent cone is always a subset of the corresponding linearized cone, we clearly have $\mathcal{T}_{NLP_*(\beta_1,\beta_2)}(x^*) \subseteq \mathcal{L}_{NLP_*(\beta_1,\beta_2)}(x^*)$ for all $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$. Invoking Lemma

2.4, we therefore obtain

$$\mathcal{T}(x^*) = \bigcup_{(\beta_1,\beta_2)\in\mathcal{P}(I_{00})} \mathcal{T}_{NLP_*(\beta_1,\beta_2)}(x^*) \subseteq \bigcup_{(\beta_1,\beta_2)\in\mathcal{P}(I_{00})} \mathcal{L}_{NLP_*(\beta_1,\beta_2)}(x^*) = \mathcal{L}_{MPVC}(x^*),$$

which proves the assertion. □

In view of the last result and the definition of standard ACQ, it is very natural to define the following Abadie-type constraint qualification tailored to MPVCs.

Definition 2.6 *Let x^* be feasible for* (1)*. Then MPVC-ACQ is said to hold at x^* if the equality $\mathcal{T}(x^*) = \mathcal{L}_{MPVC}(x^*)$ holds.*

In view of Corollary 2.5, we can state the following chain of inclusions for a feasible point x^* of (1):

$$\mathcal{T}(x^*) \subseteq \mathcal{L}_{MPVC}(x^*) \subseteq \mathcal{L}(x^*).$$

This immediately implies the next result.

Corollary 2.7 *Let x^* be feasible for* (1) *such that ACQ holds. Then MPVC-ACQ is satisfied at x^*.*

The following counterexample shows that the converse of Corollary 2.7 does not hold in general.

Example 2.8 Consider the MPVC

$$\begin{array}{ll} \min & f(x) := x_1^2 + x_2^2 \\ \text{s.t.} & H_1(x) := x_1 + x_2 \geq 0, \\ & G_1(x)H_1(x) := x_1(x_1 + x_2) \leq 0. \end{array}$$

Its solution is obviously given by $x^* := (0,0)^T$, hence we have $I_{00} = \{1\}$. The tangent cone at x^* is easily seen to be equal to the feasible set X, whereas Lemma 2.3 implies that the linearized cone is given by $\mathcal{L}(x^*) = \{d \in \mathbb{R}^2 \,|\, d_1 + d_2 \geq 0\}$. Moreover, (9) gives $\mathcal{L}_{MPVC}(x^*) = \{d \in \mathbb{R}^2 \,|\, d_1 + d_2 \geq 0, d_1(d_1 + d_2) \leq 0\}$, and this is equal to the feasible set X. Hence it follows that MPVC-ACQ holds, whereas standard ACQ is violated.

3 Necessary Conditions for Abadie CQ

The Abadie constraint qualification requires that the tangent cone $\mathcal{T}(x^*)$ is equal to the linearized cone $\mathcal{L}(x^*)$. Hence a necessary condition for the ACQ to be satisfied is that $\mathcal{T}(x^*)$ is a polyhedral convex cone. The aim of this section is to provide several characterizations of this necessary condition. To this end, we first state the following assumption.

(A1) The standard ACQ is satisfied for all nonlinear programs $\mathrm{NLP}_*(\beta_1, \beta_2)$, $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$, where x^* denotes a given feasible point of the MPVC.

In view of Lemma 2.4, it follows that (A1) implies MPVC-ACQ. Nevertheless, Assumption (A1) is still fairly weak, and a sufficient condition is the LICQ-type assumption to be introduced in Section 4, see Lemma 4.2 and its proof. Using (A1), we are able to state the following result that may be viewed as a counterpart of [14, Proposition 3] (note, however, that part of its proof is different).

Proposition 3.1 *Let $x^* \in X$ be a feasible point of the MPVC from* (1) *such that Assumption (A1) holds. Then the following statements are equivalent:*

(a) $\mathcal{T}(x^*)$ is polyhedral.

(b) $\mathcal{T}(x^*)$ is convex.

(c) For all $d^1, d^2 \in \mathcal{T}(x^*)$ and all $i \in I_{00}$, we have $\big(\nabla G_i(x^*)^T d^1\big)\big(\nabla H_i(x^*)^T d^2\big) \leq 0$.

(d) There exists a partition $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$ such that $\mathcal{T}(x^*) = \mathcal{T}_{NLP_*(\beta_1,\beta_2)}(x^*)$.

Proof. (a) $\Longrightarrow$ (b): This is obvious.

(b) $\Longrightarrow$ (c): Let $d^1, d^2 \in \mathcal{T}(x^*)$ and $i \in I_{00}$ be arbitrarily given. Define $d(\lambda) := \lambda d^1 + (1-\lambda)d^2$ for $\lambda \in (0,1)$. In view of (b), we have $d(\lambda) \in \mathcal{T}(x^*)$ for all $\lambda \in (0,1)$. Because of (A1) and Lemma 2.4, however, we have $\mathcal{T}(x^*) = \mathcal{L}_{MPVC}(x^*)$. This implies $d(\lambda) \in \mathcal{L}_{MPVC}(x^*)$ for all $\lambda \in (0,1)$. In particular, we therefore have

$$\big(\nabla G_i(x^*)^T d(\lambda)\big)\big(\nabla H_i(x^*)^T d(\lambda)\big) \leq 0.$$

Using the definition of $d(\lambda)$, this can be rewritten as

$$\begin{aligned} 0 \;\geq\; & \lambda^2\big[\big(\nabla G_i(x^*)^T d^1\big)\big(\nabla H_i(x^*)^T d^1\big)\big] \\ & +(1-\lambda)^2\big[\big(\nabla G_i(x^*)^T d^2\big)\big(\nabla H_i(x^*)^T d^2\big)\big] \\ & +\lambda(1-\lambda)\big[\big(\nabla G_i(x^*)^T d^1\big)\big(\nabla H_i(x^*)^T d^2\big) + \big(\nabla G_i(x^*)^T d^2\big)\big(\nabla H_i(x^*)^T d^1\big)\big]. \end{aligned} \tag{10}$$

Now suppose that $\big(\nabla G_i(x^*)^T d^1\big)\big(\nabla H_i(x^*)^T d^2\big) > 0$ (the case with d^1, d^2 being exchanged can be treated in a similar way). Since $d^2 \in \mathcal{T}(x^*) = \mathcal{L}_{MPVC}(x^*)$ and $i \in I_{00}$, we have $\nabla H_i(x^*)^T d^2 \geq 0$. This therefore implies $\nabla G_i(x^*)^T d^1 > 0$ and $\nabla H_i(x^*)^T d^2 > 0$. Again exploiting the fact that d^1, d^2 belong to the cone $\mathcal{L}_{MPVC}(x^*)$, we obtain $\nabla G_i(x^*)^T d^2 \leq 0$ and $\nabla H_i(x^*)^T d^1 = 0$. Taking this into account, dividing (10) by $1-\lambda$, and then letting $\lambda \to 1^-$, we get the contradiction $\big(\nabla G_i(x^*)^T d^1\big)\big(\nabla H_i(x^*)^T d^2\big) \leq 0$ from (10).

(c) $\Longrightarrow$ (d): Let (c) hold and recall that $\mathcal{T}(x^*) = \mathcal{L}_{MPVC}(x^*)$. Recall further that the cone $\mathcal{L}_{MPVC}(x^*)$ is defined by the following set of equations and inequalities:

$$\begin{array}{ll} \nabla g_i(x^*)^T d \leq 0 & (i \in I_g), \\ \nabla h_j(x^*)^T d = 0 & (j = 1, \ldots, p), \\ \nabla H_i(x^*)^T d = 0 & (i \in I_{0+}), \\ \nabla H_i(x^*)^T d \geq 0 & (i \in I_{00} \cup I_{0-}), \\ \nabla G_i(x^*)^T d \leq 0 & (i \in I_{+0}), \\ (\nabla H_i(x^*)^T d)(\nabla G_i(x^*)^T d) \leq 0 & (i \in I_{00}). \end{array} \tag{11}$$

Now let $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$ be a particular partition defined as follows: β_1 contains all the indices $i \in I_{00}$ such that there is a vector $d = d^{(i)}$ which satisfies the system (11) such that, in addition, it holds that $\nabla H_i(x^*)^T d > 0$, i.e., this inequality is satisfied strictly. Then let $\beta_2 := I_{00} \setminus \beta_1$ consist of the remaining indices. Then, for all $i \in \beta_2$ and all vectors d satisfying the system (11), we necessarily have $\nabla H_i(x^*)^T d = 0$. We now claim that $\big(\mathcal{T}(x^*) = \big)\ \mathcal{L}_{MPVC}(x^*) = \mathcal{L}_{NLP_*(\beta_1,\beta_2)}(x^*)\ \big(= \mathcal{T}_{NLP_*(\beta_1,\beta_2)}(x^*)$ in view of (A1) $\big)$. Comparing the definitions of the two cones $\mathcal{L}_{MPVC}(x^*)$ and $\mathcal{L}_{NLP_*(\beta_1,\beta_2)}(x^*)$, we only have to verify that $\nabla H_i(x^*)^T d = 0$ for all $i \in \beta_2$ and $\nabla G_i(x^*)^T d \leq 0$ for all $i \in \beta_1$. The former is true in view of our previous comments, and the latter follows from the definition of β_1 which says that, for any $i \in \beta_1$, we can find a particular vector $\tilde{d}$ satisfying the whole system (11) such that, in addition, $\nabla H_i(x^*)^T \tilde{d} > 0$. Assumption (c) then implies the desired inequality $\nabla G_i(x^*)^T d \leq 0$.

(d) $\Longrightarrow$ (a): This follows immediately from Assumption (A1). $\square$

At this point, we would like to point out that the statements (a)–(d) from Proposition 3.1 are only necessary but not sufficient conditions for ACQ. In fact, it is known, see [1] for a simple standard optimization example, that the tangent cone $\mathcal{T}(x^*)$ might be polyhedral without being equal to its linearized cone $\mathcal{L}(x^*)$.

For MPVCs, however, the situation is even more complicated since Lemma 2.4 tells us that the tangent cone $\mathcal{T}(x^*)$ is typically the union of finitely many cones. Consequently, the tangent cone $\mathcal{T}(x^*)$ is usually a nonconvex cone, i.e., the Abadie constraint qualification does not hold.

4 Sufficient Conditions for Guignard CQ

Our goal is to provide MPVC-tailored constraint qualifications which are sufficient conditions for GCQ. Since it is well-known, see, e.g., [9], that GCQ implies the KKT conditions as a necessary optimality criterion at a local minimizer of a standard optimization problem, we hereby obtain constraint qualifications to imply the KKT conditions of the MPVC, and which have a much better chance to be satisfied for MPVCs in opposite to standard constraint qualifications like LICQ or MFCQ (see the corresponding discussion in [2]).

The aim of this section is to show that GCQ holds at a feasible point of an MPVC under the presence of an LICQ-type constraint qualification which already occurred in the context of MPVCs. More precisely, the constraint qualification that we apply here was used as an assumption in [2, Corollary 2], and we formally introduce it in the following definition (the name MPVC-LICQ, however, was not used in [2]).

Definition 4.1 *We say that MPVC-LICQ is satisfied at a feasible point x^* of (1) if the gradients*

$$\begin{aligned} &\nabla h_j(x^*) && (j = 1, \ldots, p),\\ &\nabla g_i(x^*) && (i \in I_g),\\ &\nabla H_i(x^*) && (i \in I_0),\\ &\nabla G_i(x^*) && (i \in I_{00} \cup I_{+0}), \end{aligned}$$

are linearly independent.

With this notation, we can state the following result.

Lemma 4.2 *Let x^* be feasible for the MPVC (1) such that MPVC-LICQ holds at x^*. Then standard LICQ holds at x^* for all programs $NLP_*(\beta_1, \beta_2)$ with $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$ arbitrary.*

Proof. Let $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$ be given. In view of the definition of $NLP_*(\beta_1, \beta_2)$ in (7), we have to show that the gradients

$$\begin{aligned} &\nabla h_j(x^*) && (j = 1, \ldots, p),\\ &\nabla g_i(x^*) && (i \in I_g),\\ &\nabla H_i(x^*) && (i \in I_0),\\ &\nabla G_i(x^*) && (i \in \beta_1 \cup I_{+0}), \end{aligned}$$

are linearly independent. Since we have $\beta_1 \subseteq I_{00}$, this is trivially satisfied, because MPVC-LICQ holds. □

We are now in a position to state the first main result which gives a relatively simple sufficient condition for the Guignard CQ to be satisfied.

Theorem 4.3 *Let x^* be feasible for the MPVC (1) such that MPVC-LICQ is satisfied at x^*. Then GCQ holds at x^*.*

Proof. In view of Definition 2.2 and the well-known inclusion $\mathcal{L}(x^*)^* \subseteq \mathcal{T}(x^*)^*$, we only need to prove that the converse inclusion $\mathcal{T}(x^*)^* \subseteq \mathcal{L}(x^*)^*$ holds. To this end, first recall that we have

$$\mathcal{T}(x^*) = \bigcup_{(\beta_1,\beta_2)\in\mathcal{P}(I_{00})} \mathcal{T}_{NLP_*(\beta_1,\beta_2)}(x^*)$$

in view of Lemma 2.4 (a). Invoking [3, Theorem 3.1.9] therefore yields

$$\mathcal{T}(x^*)^* = \bigcap_{(\beta_1,\beta_2)\in\mathcal{P}(I_{00})} \mathcal{T}_{NLP_*(\beta_1,\beta_2)}(x^*)^*. \tag{12}$$

Since MPVC-LICQ holds at x^* for (1), we know by Lemma 4.2 that LICQ and thus ACQ are satisfied at x^* for $NLP_*(\beta_1,\beta_2)$ and for all $(\beta_1,\beta_2)\in\mathcal{P}(I_{00})$. Hence, we have $\mathcal{T}_{NLP_*(\beta_1,\beta_2)}(x^*) = \mathcal{L}_{NLP_*(\beta_1,\beta_2)}(x^*)$ for all $(\beta_1,\beta_2)\in\mathcal{P}(I_{00})$. Recalling the representation of $\mathcal{L}_{NLP_*(\beta_1,\beta_2)}(x^*)$ from (8), and using [3, Theorem 3.2.2], we obtain

$$\begin{aligned}&\mathcal{L}_{NLP_*(\beta_1,\beta_2)}(x^*)^* = \\ &\big\{v\in\mathbb{R}^n \mid v = -\sum_{i\in I_g}\mu_i^g\nabla g_i(x^*) - \sum_{j=1,\dots,p}\mu_j^h\nabla h_j(x^*) + \sum_{i\in I_0}\mu_i^H\nabla H_i(x^*) - \sum_{i\in I_{+0}\cup\beta_1}\mu_i^G\nabla G_i(x^*) \\ &\qquad \text{with}\quad \mu_i^g\geq 0\ (i\in I_g),\quad \mu_i^H\geq 0\ (i\in I_{0-}\cup\beta_1),\quad \mu_i^G\geq 0\ (i\in I_{+0}\cup\beta_1)\big\}.\end{aligned}$$

In a similar way, we obtain

$$\begin{aligned}&\mathcal{L}(x^*)^* = \\ &\big\{v\in\mathbb{R}^n \mid v = -\sum_{i\in I_g}\mu_i^g\nabla g_i(x^*) - \sum_{j=1,\dots,p}\mu_j^h\nabla h_j(x^*) + \sum_{i\in I_0}\mu_i^H\nabla H_i(x^*) - \sum_{i\in I_{+0}}\mu_i^G\nabla G_i(x^*) \\ &\qquad \text{with}\quad \mu_i^g\geq 0\ (i\in I_g),\quad \mu_i^H\geq 0\ (i\in I_{0-}\cup I_{00}),\quad \mu_i^G\geq 0\ (i\in I_{+0})\big\}.\end{aligned}$$

Now let $v\in\mathcal{T}(x^*)^* = \bigcap_{(\beta_1,\beta_2)\in\mathcal{P}(I_{00})} \mathcal{L}_{NLP_*(\beta_1,\beta_2)}(x^*)^*$. Moreover, choose $(\beta_1,\beta_2)\in\mathcal{P}(I_{00})$ arbitrarily and put $(\tilde\beta_1,\tilde\beta_2) := (\beta_2,\beta_1)$. Using the above representation of $\mathcal{L}_{NLP_*(\beta_1,\beta_2)}(x^*)^*$, it follows that there exists a vector $\mu = (\mu^g,\mu^h,\mu^H,\mu^G)$ with

$$\mu_i^g\geq 0\ (i\in I_g),\ \ \mu_i^H\geq 0\ (i\in I_{0-}\cup\beta_1),\ \ \mu_i^G\geq 0\ (i\in I_{+0}\cup\beta_1) \tag{13}$$

such that

$$v = -\sum_{i\in I_g}\mu_i^g\nabla g_i(x^*) - \sum_{j=1,\dots,p}\mu_j^h\nabla h_j(x^*) + \sum_{i\in\beta_1\cup\beta_2\cup I_{0-}\cup I_{0+}}\mu_i^H\nabla H_i(x^*) - \sum_{i\in I_{+0}\cup\beta_1}\mu_i^G\nabla G_i(x^*). \tag{14}$$

However, since v also belongs to $\mathcal{L}_{NLP_*(\tilde\beta_1,\tilde\beta_2)}(x^*)^*$, we obtain in a similar way the existence of a certain vector $\tilde\mu = (\tilde\mu^g,\tilde\mu^h,\tilde\mu^H,\tilde\mu^G)$ satisfying

$$\tilde\mu_i^g\geq 0\ (i\in I_g),\ \ \tilde\mu_i^H\geq 0\ (i\in I_{0-}\cup\tilde\beta_1),\ \ \tilde\mu_i^G\geq 0\ (i\in I_{+0}\cup\tilde\beta_1)$$

such that

$$v = -\sum_{i\in I_g}\tilde\mu_i^g\nabla g_i(x^*) - \sum_{j=1,\dots,p}\tilde\mu_j^h\nabla h_j(x^*) + \sum_{i\in\tilde\beta_1\cup\tilde\beta_2\cup I_{0-}\cup I_{0+}}\tilde\mu_i^H\nabla H_i(x^*) - \sum_{i\in I_{+0}\cup\tilde\beta_1}\tilde\mu_i^G\nabla G_i(x^*). \tag{15}$$

Subtracting the two representations (14) and (15) of v from each other, we obtain

$$\begin{aligned}0 = &-\sum_{i\in I_g}(\mu_i^g - \tilde{\mu}_i^g)\nabla g_i(x^*) - \sum_{j=1,\ldots,p}(\mu_j^h - \tilde{\mu}_j^h)\nabla h_j(x^*) + \sum_{i\in I_{0-}\cup I_{0+}}(\mu_i^H - \tilde{\mu}_i^H)\nabla H_i(x^*)\\ &+ \sum_{i\in\beta_1(=\tilde{\beta}_2)}(\mu_i^H - \tilde{\mu}_i^H)\nabla H_i(x^*) + \sum_{i\in\beta_2(=\tilde{\beta}_1)}(\mu_i^H - \tilde{\mu}_i^H)\nabla H_i(x^*) - \sum_{i\in\beta_1}\mu_i^G\nabla G_i(x^*)\\ &+ \sum_{i\in\beta_2(=\tilde{\beta}_1)}\tilde{\mu}_i^G\nabla G_i(x^*) - \sum_{i\in I_{+0}}(\mu_i^G - \tilde{\mu}_i^G)\nabla G_i(x^*).\end{aligned}$$

Since MPVC-LICQ holds at x^*, all gradients occuring in the previous formula are linearly independent. Consequently, all coefficients are zero. In particular, we obtain $\mu_i^H = \tilde{\mu}_i^H \geq 0$ $(i \in \beta_2)$ and $\mu_i^G = 0$ $(i \in \beta_1)$. Taking this into account and using (14), (13), we obtain the representation

$$v = -\sum_{i\in I_g}\mu_i^g\nabla g_i(x^*) - \sum_{j=1,\ldots,p}\mu_j^h\nabla h_j(x^*) + \sum_{i\in I_0}\mu_i^H\nabla H_i(x^*) - \sum_{i\in I_{+0}}\mu_i^G\nabla G_i(x^*)$$

with

$$\mu_i^g \geq 0\ (i \in I_g), \quad \mu_i^H \geq 0\ (i \in I_{0-} \cup I_{00}), \quad \mu_i^G \geq 0\ (i \in I_{+0}).$$

This shows that v belongs to $\mathcal{L}(x^*)^*$, cf. the above representation of this dual cone. □

In contrast to Theorem 4.3, the following counterexample shows that MPVC-LICQ does, in general, not imply standard ACQ.

Example 4.4 Consider the particular MPVC from Example 2.8. Here the gradients

$$\nabla G_1(x^*) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \nabla H_1(x^*) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

are linearly independent, showing that MPVC-LICQ holds at the solution x^*, whereas it was already noted in Example 2.8 that standard ACQ is violated.

The following result is an immediate consequence of Theorem 4.3.

Corollary 4.5 *Let x^* be a local minimizer of (1) such that MPVC-LICQ holds at x^*. Then there exist unique Lagrange multipliers $\lambda_i \in \mathbb{R}\,(i = 1,\ldots,m), \mu_j \in \mathbb{R}\,(j = 1,\ldots,p), \eta_i^H, \eta_i^G \in \mathbb{R}\,(i = 1,\ldots,l)$ such that*

$$\nabla f(x^*) + \sum_{i=1}^{m}\lambda_i\nabla g_i(x^*) + \sum_{j=1}^{p}\mu_j\nabla h_j(x^*) - \sum_{i=1}^{l}\eta_i^H\nabla H_i(x^*) + \sum_{i=1}^{l}\eta_i^G\nabla G_i(x^*) = 0 \quad (16)$$

and

$$\begin{aligned}&h_j(x^*) = 0 \quad \forall j = 1,\ldots,p,\\ &\lambda_i \geq 0,\ \ g_i(x^*) \leq 0,\ \ \lambda_i g_i(x^*) = 0 \quad \forall i = 1,\ldots,m,\\ &\eta_i^H = 0\,(i \in I_+),\ \ \eta_i^H \geq 0\,(i \in I_{00} \cup I_{0-}),\ \ \eta_i^H \text{ free } (i \in I_{0+}),\\ &\eta_i^G = 0\,(i \in I_0 \cup I_{+-}),\ \ \eta_i^G \geq 0\,(i \in I_{+0}).\end{aligned} \quad (17)$$

Proof. By Theorem 4.3, we know that MPVC-LICQ implies GCQ. It is well-known, however, that GCQ implies the KKT conditions as necessary optimality conditions, see, e.g., [3]. Since [2, Theorem 1] shows that (16) and (17) are precisely the KKT conditions of (1), the statement follows (with the uniqueness of the multipliers being an immediate consequence of the linear independence of the gradients occuring in MPVC-LICQ). □

We would like to emphasize that the previous result is an improvement of [2, Corollary 2], where it was shown that MPVC-LICQ implies slightly weaker optimality conditions at a local minimizer.

To illustrate Corollary 4.5, consider once again the particular MPVC from Example 2.8. This example satisfies MPVC-LICQ at its solution $x^* := (0,0)^T$, see Example 4.4. An easy calculation shows that $\eta_1^G := 0$ and $\eta_1^H := 0$ are the unique Lagrange multipliers associated with x^* such that the KKT conditions from (16) and (17) hold.

5 Improved Sufficient Conditions for Guignard CQ

The MPVC-LICQ condition from the previous section gives a relatively simple LICQ-type condition which guarantees that GCQ holds and, therefore, the standard KKT conditions represent first order optimality conditions for our MPVC. However, it is possible to relax this assumption. The approach we follow here is motivated by the corresponding analysis given in [14, 6] for MPECs and is based on the notion of a singular inequality in the context of a linear system described by linear equations and linear inequalities, see [17].

Definition 5.1 *Given the linear system*

$$Ax \leq b, \quad Cx = d, \tag{18}$$

an inequality $a_i x \leq b_i$ is said to be nonsingular if there exists a feasible solution of the system (18) *which satisfies this inequality strictly; otherwise it is called singular. Here, a_i denotes the i-th row of the matrix A.*

Implicitly, the notion of a nonsingular inequality was already used in the proof of Proposition 3.1, however, there nonsingular inequalities were used in the context of a nonlinear system of equations and inequalities, whereas here we use this notion only for a linear system (as given in [17], note that a formal definition for a general nonlinear system would be much more complicated).

We now apply nonsingularity to the linearized cone $\mathcal{L}(x^*)$: Let β^H denote the subset of I_{00} consisting of all indices i such that the inequality $\nabla H_i(x^*)^T d \geq 0$ is nonsingular in the system defining $\mathcal{L}(x^*)$. Thus, we obtain the refined representation

$$\begin{aligned} \mathcal{L}(x^*) = \big\{ d \in \mathbb{R}^n \mid\ & \nabla g_i(x^*)^T d \leq 0 \quad (i \in I_g), \\ & \nabla h_j(x^*)^T d = 0 \quad (j = 1, \ldots, p), \\ & \nabla H_i(x^*)^T d = 0 \quad (i \in I_{0+} \cup I_{00} \setminus \beta^H), \\ & \nabla H_i(x^*)^T d \geq 0 \quad (i \in I_{0-} \cup \beta^H), \\ & \nabla G_i(x^*)^T d \leq 0 \quad (i \in I_{+0}) \big\}. \end{aligned} \tag{19}$$

In view of [3, Theorem 3.2.2], the dual is therefore given by

$$\mathcal{L}(x^*)^* = \Big\{v \in \mathbb{R}^n \mid v = -\sum_{i\in I_g} \mu_i^g \nabla g_i(x^*) - \sum_{j=1,\ldots,p} \mu_j^h \nabla h_j(x^*) + \sum_{i\in I_0} \mu_i^H \nabla H_i(x^*) - \sum_{i\in I_{+0}} \mu_i^G \nabla G_i(x^*)$$
$$\text{with} \quad \mu_i^g \geq 0 \ (i \in I_g), \quad \mu_i^H \geq 0 \ (i \in I_{0-} \cup \beta^H), \quad \mu_i^G \geq 0 \ (i \in I_{+0})\Big\}. \quad (20)$$

We will now use the index set β^H to define the following assumption which occurs similarly in the context of MPECs, cf. [14, 6].

(A2) For each $i_0 \in \beta^H$ there exists a vector d such that

$$\begin{array}{ll} \nabla G_{i_0}(x^*)^T d < 0, & \\ \nabla g_i(x^*)^T d = 0 & (i \in I_g), \\ \nabla h_j(x^*)^T d = 0 & (j = 1, \ldots, p), \\ \nabla H_i(x^*)^T d = 0 & (i \in I_0), \\ \nabla G_i(x^*)^T d = 0 & (i \in I_{+0} \cup I_{00} \setminus \{i_0\}), \end{array} \quad (21)$$

where x^* is a given feasible point of (1).

In the next result, we will present an equivalent formulation of (A2) which will be used in the proof of our upcoming main theorem. Thus, we might as well have omitted (A2), but it seemed somewhat appealing to us not to leave it out in order to stress the close relation to the approach for MPECs, made in the above mentioned papers.

Lemma 5.2 *Let x^* be feasible for (1), and let β^H be defined as above. Then the following statements are equivalent:*

(a) The following implication holds:

$$0 = \sum_{i\in I_g} \mu_i^g \nabla g_i(x^*) + \sum_{j=1}^{p} \mu_j^h \nabla h_j(x^*) + \sum_{i\in I_0} \mu_i^H \nabla H_i(x^*) + \sum_{i\in I_{+0}\cup I_{00}} \mu_i^G \nabla G_i(x^*)$$
$$\Longrightarrow \quad \mu_i^G = 0 \quad (i \in \beta^H). \quad (22)$$

(b) (A2) holds.

Proof. (a) $\Longrightarrow$ (b): Let (a) be satisfied and suppose that (b) does not hold. Then there exists an index $i_0 \in \beta^H$ such that the corresponding linear system (21) has no solution. According to Motzkin's theorem of the alternative (see, e.g., [11]), we therefore get a number $u_{i_0} > 0$ and a vector μ, whose components are denoted in a way compatible to the system (21), with

$$0 = u_{i_0}\nabla G_{i_0}(x^*)+\sum_{i\in I_g}\mu_i^g\nabla g_i(x^*)+\sum_{j=1}^{p}\mu_j^h\nabla h_j(x^*)+\sum_{i\in I_0}\mu_i^H\nabla H_i(x^*)+\sum_{i\in I_{+0}\cup I_{00}\setminus\{i_0\}}\mu_i^G\nabla G_i(x^*).$$

Using (a), however, we in particular get $u_{i_0} = 0$, a contradiction to $u_{i_0} > 0$. Hence (A2) is satisfied.

(b) $\Longrightarrow$ (a): Let (b) be satisfied and suppose that (a) does not hold. Then there is a vector $\mu = (\mu^g, \mu^h, \mu^G, \mu^H)$ with

$$0 = \sum_{i\in I_g}\mu_i^g\nabla g_i(x^*) + \sum_{j=1}^{p}\mu_j^h\nabla h_j(x^*) + \sum_{i\in I_0}\mu_i^H\nabla H_i(x^*) + \sum_{i\in I_{+0}\cup I_{00}}\mu_i^G\nabla G_i(x^*)$$

such that $\mu_{i_0}^G \neq 0$ for some index $i_0 \in \beta^H$. Reordering the components of μ, this means that $\mu \in \mathrm{Null}(A)$, the null space of A, where A is defined by

$$A^T := \begin{pmatrix} \nabla G_{i_0}(x^*)^T & \\ \nabla g_i(x^*)^T & (i \in I_g) \\ \nabla h_j(x^*)^T & (j = 1, \ldots, p) \\ \nabla H_i(x^*)^T & (i \in I_0) \\ \nabla G_i(x^*)^T & (i \in I_{+0} \cup I_{00} \setminus \{i_0\}) \end{pmatrix}.$$

On the other hand, Assumption (A2) shows that there is a vector d and a scalar $\tau \neq 0$ such that $A^T d = (\tau, 0, \ldots, 0)^T =: x$. This implies $\mu^T x = \mu_{i_0}\tau \neq 0$. On the other hand, we have $\mu^T x = \mu^T A^T d = (A\mu)^T d = 0$ since $\mu \in \mathrm{Null}(A)$. This contradiction shows that (a) holds. □

We now come to the main result of this section which says that MPVC-ACQ (see Definition 2.6) together with (A2) implies GCQ.

Theorem 5.3 *Let x^* be feasible for* (1) *such that MPVC-ACQ and (A2) hold at x^*. Then GCQ is satisfied at x^*.*

Proof. As in the proof of Theorem 4.3, we only need to show $\mathcal{T}(x^*)^* \subseteq \mathcal{L}(x^*)^*$. Since MPVC-ACQ holds at x^*, we have

$$\mathcal{T}(x^*) = \mathcal{L}_{MPVC}(x^*) \overset{\text{Lemma 2.4}}{=} \bigcup_{(\beta_1,\beta_2)\in\mathcal{P}(I_{00})} \mathcal{L}_{NLP_*(\beta_1,\beta_2)}(x^*).$$

Dualizing yields

$$\mathcal{T}(x^*)^* = \bigcap_{(\beta_1,\beta_2)\in\mathcal{P}(I_{00})} \mathcal{L}_{NLP_*(\beta_1,\beta_2)}(x^*)^*. \tag{23}$$

Now take $v \in \mathcal{T}(x^*)^*$ arbitrarily, then we have $v \in \mathcal{L}_{NLP_*(\beta_1,\beta_2)}(x^*)^*$ for all $(\beta_1, \beta_2) \in \mathcal{P}(I_{00})$ because of (23). Now put

$$\hat{\beta}_1 := \beta^H \quad \text{and} \quad \hat{\beta}_2 := I_{00} \setminus \hat{\beta}_1, \tag{24}$$

and define

$$(\tilde{\beta}_1, \tilde{\beta}_2) := (\hat{\beta}_2, \hat{\beta}_1).$$

Using a similar reasoning as in the proof of Theorem 4.3, we therefore get two representations

$$v = -\sum_{i \in I_g} \hat{\mu}_i^g \nabla g_i(x^*) - \sum_{j=1,\ldots,p} \hat{\mu}_j^h \nabla h_j(x^*) + \sum_{i \in I_0} \hat{\mu}_i^H \nabla H_i(x^*) - \sum_{i \in I_{+0} \cup \hat{\beta}_1} \hat{\mu}_i^G \nabla G_i(x^*) \tag{25}$$

and

$$v = -\sum_{i \in I_g} \tilde{\mu}_i^g \nabla g_i(x^*) - \sum_{j=1,\ldots,p} \tilde{\mu}_j^h \nabla h_j(x^*) + \sum_{i \in I_0} \tilde{\mu}_i^H \nabla H_i(x^*) - \sum_{i \in I_{+0} \cup \tilde{\beta}_1} \tilde{\mu}_i^G \nabla G_i(x^*)$$

with the coefficients satisfying, in particular,

$$\hat{\mu}_i^g \geq 0 \ (i \in I_g), \quad \hat{\mu}_i^H \geq 0 \ (i \in I_{0-} \cup \hat{\beta}_1), \quad \hat{\mu}_i^G \geq 0 \ (i \in I_{+0} \cup \hat{\beta}_1). \tag{26}$$

Subtracting the two representations of v from each other, we get

$$\begin{aligned} 0 = \; - & \sum_{i \in I_g} (\hat{\mu}_i^g - \tilde{\mu}_i^g) \nabla g_i(x^*) - \sum_{j=1}^{p} (\hat{\mu}_j^h - \tilde{\mu}_j^h) \nabla h_j(x^*) + \sum_{i \in I_{0-} \cup I_{0+}} (\hat{\mu}_i^H - \tilde{\mu}_i^H) \nabla H_i(x^*) \\ & + \sum_{i \in I_{00} (= \hat{\beta}_1 \cup \tilde{\beta}_1)} (\hat{\mu}_i^H - \tilde{\mu}_i^H) \nabla H_i(x^*) - \sum_{i \in I_{+0}} (\hat{\mu_i}^G - \tilde{\mu}_i^G) \nabla G_i(x^*) - \sum_{i \in \hat{\beta}_1} \hat{\mu}_i^G \nabla G_i(x^*) \\ & + \sum_{i \in \hat{\beta}_2 (= \tilde{\beta}_1)} \tilde{\mu}_i^G \nabla G_i(x^*). \end{aligned} \tag{27}$$

Since (A2) holds, we know by Lemma 5.2 that (27) implies

$$\hat{\mu}_i^G = 0 \ (i \in \hat{\beta}_1 = \beta^H).$$

Taking into account (25) and (26), we therefore have

$$v = -\sum_{i \in I_g} \hat{\mu}_i^g \nabla g_i(x^*) - \sum_{j=1,\ldots,p} \hat{\mu}_j^h \nabla h_j(x^*) + \sum_{i \in I_0} \hat{\mu}_i^H \nabla H_i(x^*) - \sum_{i \in I_{+0}} \hat{\mu}_i^G \nabla G_i(x^*)$$

with $\hat{\mu}_i^g \geq 0 \ (i \in I_g), \hat{\mu}_i^H \geq 0 \ (i \in I_{0-} \cup \hat{\beta}_1)$ and $\hat{\mu}_i^G \geq 0 \ (i \in I_{+0})$. Using the definition of $\hat{\beta}_1$ from (24), it follows from the representation of $\mathcal{L}(x^*)^*$ given in (20) that v is an element of this dual cone. □

To state another sufficient condition for GCQ, we have to introduce the notion of *partial MPVC-LICQ*. It is motivated by a corresponding concept introduced for MPECs in [18] and successfully used, e.g., in [19, 7].

Definition 5.4 *Let x^* be feasible for* (1)*. Partial MPVC-LICQ is said to hold at x^* if for any vector $\mu := (\mu^g, \mu^h, \mu^H, \mu^G)$ with*

$$0 = \sum_{i \in I_g} \mu_i^g \nabla g_i(x^*) + \sum_{j=1}^{p} \mu_j^h \nabla h_j(x^*) + \sum_{i \in I_0} \mu_i^H \nabla H_i(x^*) + \sum_{i \in I_{+0} \cup I_{00}} \mu_i^G \nabla G_i(x^*)$$

it follows that $\mu_i^G = 0$ for all $i \in I_{00}$.

Corollary 5.5 *Let x^* be feasible for* (1) *such that MPVC-ACQ and partial MPVC-LICQ holds. Then GCQ is satisfied at x^*. In particular, there exist Lagrange multipliers μ such that the KKT conditions from* (16) *and* (17) *hold.*

Proof. Since partial MPVC-LICQ implies statement (a) from Lemma 5.2 and thus (A2), the assertion follows from Theorem 5.3. □

Note that the assumptions of Theorem 5.3 and Corollary 5.5 are weaker than the MPVC-LICQ condition used in Theorem 4.3. In fact, MPVC-LICQ obviously implies both (A2) as well as partial MPVC-LICQ. Moreover, MPVC-LICQ also implies MPVC-ACQ since, under MPVC-LICQ, each nonlinear program $\mathrm{NLP}_*(\beta_1, \beta_2)$ satisfies standard LICQ, hence standard ACQ, therefore (A1) holds which, in view of Lemma 2.4, is a sufficient condition for MPVC-ACQ. The fact that both (A2) and MPVC-ACQ are indeed strictly weaker than MPVC-LICQ is illustrated by the following example.

Example 5.6 Consider the MPVC

$$\begin{array}{ll} \min & f(x) := x_1^2 + x_2^2 + x_3^2 + x_4^2 \\ \text{s.t.} & g_1(x) := -x_1 \le 0, \\ & g_2(x) := -x_2 \le 0, \\ & g_3(x) := x_1 - x_2 \le 0, \\ & H_1(x) := x_3 \ge 0, \\ & G_1(x) H_1(x) := x_4 x_3 \le 0. \end{array} \tag{28}$$

The unique solution of (28) is $x^* := (0,0,0,0)^T$. Thus, we have $I_g = \{1,2,3\}$ and $I_{00} = \{1\}$. It is easy to see that MPVC-LICQ is violated, since the gradients $\nabla g_1(x^*) = (-1,0,0,0)^T$, $\nabla g_2(x^*) = (0,-1,0,0)^T$ and $\nabla g_3(x^*) = (1,-1,0,0)^T$ are linearly dependent.

On the other hand, partial MPVC-LICQ (and therefore also Assumption (A2)) holds since

$$0 = \mu_1^g \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mu_2^g \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \mu_3^g \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \mu_1^H \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \mu_1^G \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

implies $\mu_1^G = 0$. Furthermore, since all functions g_1, g_2, g_3, H_1, G_1 are linear, ACQ is satisfied for all $NLP_*(\beta_1, \beta_2), (\beta_1, \beta_2) \in \mathcal{P}(I_{00})$. Hence (A1) holds which, in turn, implies that MPVC-ACQ is also satisfied.

6 Final Remarks

Knowing that both the standard LICQ and MFCQ conditions are typically not satisfied for our MPVC from (1), we investigated two weaker constraint qualifications, namely ACQ and GCQ. Our results indicate that ACQ is still a very strong condition, whereas GCQ holds under relatively mild assumptions.

On the other hand, if one is interested in solving an MPVC numerically, which is one of our future research topics, the GCQ is certainly not enough in order to get a numerical stable algorithm. However, we believe that some of the sufficient conditions like the MPVC-LICQ used in Section 4, will also play an important role in the design and convergence analysis of a suitable method.

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