On finite termination of an iterative method for linear complementarity problems

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Abstract

Based on a well-known reformulation of the linear complementarity problem (LCP) as a nondifferentiable system of nonlinear equations, a Newton-type method will be described for the solution of LCPs. Under certain assumptions, it will be shown that this method has a finite termination property, i.e., if an iterate is sufficiently close to a solution of LCP, the method finds this solution in one step. This result will be applied to a recently proposed algorithm by Harker and Pang in order to prove that their algorithm also has the finite termination property.

Keywords: linear complementarity problems; nonsmooth equations; generalized Jacobians; Newton's method; finite termination

1 Introduction

In this paper, we will consider the linear complementarity problem $\mathrm{LCP}(q, M)$ of finding a vector pair $(x, y) \in \mathbb{R}^{2n}$ such that the conditions

$$x \geq 0, \quad y \geq 0, \quad x^T y = 0, \quad Mx + q = y$$

are satisfied, where the matrix $M \in \mathbb{R}^{n \times n}$ and the vector $q \in \mathbb{R}^n$ are given.

Several well-known methods for the solution of $\mathrm{LCP}(q, M)$ exist. Many of them are described in the books by Murty [23], Cottle, Pang and Stone [3], Kojima et al. [19] and Harker [11]. Here, we will focus on a Newton-type method being applied to the nonlinear system of equations $F(z) = 0$, where $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$

Preprint submitted to Elsevier Science *8 March 2005*

is defined by

$$F(z) := F(x, y) := \begin{pmatrix} Mx + q - y \\ \min\{x, y\} \end{pmatrix} \tag{1}$$

with the min-operator taken componentwise. Obviously, a vector pair (x, y) is a solution of LCP(q, M) if and only if (x, y) is a zero of F. A Newton-type method applied to this characterization was proposed by Pang [24,25] in connection with nonlinear complementarity and variational inequality problems. This approach has been specialized by Harker and Pang [12] to the linear complementarity problem. In order to overcome the nondifferentiability of the operator F, they made use of the so-called B-derivative of F. The numerical results reported by Harker and Pang [12] are quite promising. In particular, their method outdoes Lemke's classical complementary pivot algorithm if measured in CPU-times. An unsolved question raised by Harker and Pang is, however, the finite termination of their method. It is the main contribution of this paper to show that their method has indeed the finite termination property under some standard assumptions.

We want to mention some relevant works. The characterization (1) of LCP(q, M) as a system of nonlinear equations is not the only one. The first such approach can be found in Mangasarian [22], who presents a general class of characterizations. Theoretical and numerical results for particular members of Mangasarian's class can be found, e.g., in Watson [31], Subramanian [29] and Ferris and Lucidi [6]. An even more general approach is given in Kanzow [15] and Tseng [30]. Special characterizations not belonging to the Mangasarian-class are considered in Fischer [7–9], Harker and Xiao [13] and Dirkse and Ferris [4]. Chen and Harker [1] and Kanzow [17] use similar ideas in an interior-point setting.

Most of the above-mentioned papers consider nondifferentiable characterizations of the complementarity problem, although there are also some differentiable ones. Methods based on (appropriate) nondifferentiable characterizations usually seem to be numerically more successful than their differentiable counterparts. There are at least two reasons for this. First, the differentiable characterizations involve more complicated functions, i.e., the characterizations themselves are more nonlinear, and second, the Jacobian matrices of differentiable characterizations of LCP(q, M) are singular at degenerate solutions of LCP(q, M), see Theorem 3.1 and Remark 3.2 in Kanzow and Kleinmichel [18].

This paper is organized as follows. After reviewing some background material in Section 2 we will describe our algorithm. In contrast to the approach by Harker and Pang [12], this algorithm is based on Clarke's [2] generalized Jacobian in order to overcome the difficulty of nondifferentiable points. In Section 3, we will prove a finite termination property of this algorithm under certain assumptions. In Section 4 we will apply our results to a method of Harker and Pang [12] and prove, in this way, that their algorithm also has the finite termination property. Some final remarks in Section 5 will conclude this paper.

In the following, $\|z\|$ denotes the Euclidean norm of a vector z of appropriate dimension. The letter "k" is always used as an iteration index, whereas a subscript "i" usually denotes the ith component of a vector. Throughout this paper, the index set $\{1, \ldots, n\}$ is abbreviated by $\mathcal{I}$. For arbitrary $p \in \mathbb{R}^n$ and $J \subseteq \mathcal{I}$, the vector p_J consists of the components $p_i, i \in J$. Similarly, for a given matrix $M \in \mathbb{R}^{n\times n}$, M_{JJ} denotes the submatrix $(m_{ij})_{i,j\in J}$. Moreover, $M_{i\cdot}$ means the ith row vector of M. The n-vector of all ones is denoted by e.

2 Mathematical Background and Algorithm

In this section, we will restate some basic definitions which will be used in the subsequent analysis.

Definition 1 *Let $M \in \mathbb{R}^{n\times n}$. Then M is said to be a*

(a) nondegenerate matrix *if $det(M_{JJ}) \neq 0$ for all $J \subseteq \mathcal{I}$;*
(b) P-matrix *if $det(M_{JJ}) > 0$ for all $J \subseteq \mathcal{I}$.*

Obviously, any P-matrix is nondegenerate. Moreover, it is well-known that M is a P-matrix if and only if, for all $x \in \mathbb{R}^n, x \neq 0$, an index $i \in \mathcal{I}$ exists such that $x_i \neq 0$ and $x_i[Mx]_i > 0$, see [3].

Let $z^* := (x^*, y^*) \in \mathbb{R}^{2n}$ be a solution of LCP(q, M), and define the following index sets:

$$\begin{aligned} \alpha &:= \alpha(z^*) := \{i \in \mathcal{I} \mid x_i^* > 0 = y_i^*\}, \\ \beta &:= \beta(z^*) := \{i \in \mathcal{I} \mid x_i^* = 0 = y_i^*\}, \\ \gamma &:= \gamma(z^*) := \{i \in \mathcal{I} \mid x_i^* = 0 < y_i^*\}. \end{aligned}$$

Note that β is the set of degenerate indices, and that the results of this paper hold without the assumption that β is an empty set.

Definition 2 *A solution $z^* := (x^*, y^*) \in \mathbb{R}^{2n}$ of LCP(q, M) is called*

(a) b-regular *if the submatrices $M_{\delta\delta}$ are nonsingular for all $\alpha \subseteq \delta \subseteq \alpha \cup \beta$;*
(b) R-regular *if the submatrix $M_{\alpha\alpha}$ is nonsingular and the matrix*

$$M_{\beta\beta} - M_{\beta\alpha}M_{\alpha\alpha}^{-1}M_{\alpha\beta} \tag{2}$$

is a P-matrix.

We note that the matrix (2) is the Schur-complement of $M_{\alpha\alpha}$ in

$$\begin{pmatrix} M_{\alpha\alpha} & M_{\alpha\beta} \\ M_{\beta\alpha} & M_{\beta\beta} \end{pmatrix}.$$

Obviously, the b-regularity assumption is weaker than the R-regularity assumption. The latter originates from Robinson's concept of a strongly regular solution of a generalized equation [28]. Both assumptions are important for stability results and to prove fast local convergence of certain Newton-type methods, see, e.g., [3,5].

Now let F be the nonlinear operator defined in (1). Furthermore, let

$$\mathcal{D} := \{z = (x, y) \in \mathbb{R}^{2n} \,|\, x_i \neq y_i \text{ for all } i \in \mathcal{I}\}$$

denote the set of vectors z at which F is differentiable. Obviously, F is a locally Lipschitz-continuous operator. Hence, we can define its B-subdifferential (see [26])

$$\partial_B F(z) := \left\{ G \in \mathbb{R}^{n\times n} \,|\, \exists \{z^k\} \subseteq \mathcal{D} : \lim_{k\to\infty} z^k = z, G = \lim_{k\to\infty} \nabla F(z^k) \right\}$$

and its generalized Jacobian (see [2])

$$\partial F(z) := conv\,\{\partial_B F(z)\},$$

where $conv(\mathcal{A})$ is the convex hull of a set $\mathcal{A}$, and $\nabla F(z^k)$ denotes the Jacobian matrix of F at z^k. If $z \in \mathcal{D}$, then $\partial_B F(z) = \partial F(z) = \{\nabla F(z)\}$ is a singleton. Otherwise $\partial F(z)$ can be shown to be a nonempty, convex and compact set, see Clarke [2].

From the definitions of $F, \partial_B F(z)$ and $\partial F(z)$, we obtain that any $G \in \partial F(z)$ has the following structure:

$$G = G(a) := \begin{pmatrix} M & -I \\ D_a & I - D_a \end{pmatrix}, \qquad D_a := \text{diag}(a_1, \ldots, a_n), \tag{3}$$

where the vector $a \in \mathbb{R}^n$ with $a_i \in [0, 1]$ for $i \in \mathcal{I}$ depends on which $G \in \partial F(z)$ is chosen. More precisely, we get the following representations of the B-subdifferential and of the generalized Jacobian:

$$\partial_B F(z) = \{G(a) \,|\, a_i = 1 \text{ if } x_i < y_i,\ a_i = 0 \text{ if } x_i > y_i,\ a_i \in \{0, 1\} \text{ if } x_i = y_i\}, \tag{4}$$

$$\partial F(z) = \{G(a) \,|\, a_i = 1 \text{ if } x_i < y_i,\ a_i = 0 \text{ if } x_i > y_i,\ a_i \in [0, 1] \text{ if } x_i = y_i\}. \tag{5}$$

We will now give a formal description of an algorithm whose theoretical properties will be analyzed in the following section.

Algorithm 1 *(Nonsmooth Newton method)*

(S.0) *Choose* $z^0 := (x^0, y^0) \in \mathbb{R}^{2n}, \beta, \sigma \in (0,1)$ *and set* $k := 0$.
(S.1) *If* $\|F(z^k)\| = 0$, *stop. (*z^k *solves the LCP*(q, M)*.)*
(S.2) *Choose a nonsingular matrix* $G_k \in \partial F(z^k)$ *and compute* $\Delta z^k := (\Delta x^k, \Delta y^k)$ *as the unique solution of the generalized Newton equation*

$$G_k \Delta z = -F(z^k). \tag{6}$$

(S.3) *Set* $t_k := \beta^{m_k}$, *where* m_k *is the smallest nonnegative integer* m *satisfying the Armijo condition*

$$\|F(z^k + \beta^m \Delta z^k)\|^2 \leq (1 - \beta^m \sigma)\|F(z^k)\|^2.$$

(S.4) *Set* $z^{k+1} := z^k + t_k \Delta z^k,\ k := k + 1$ *and go to (S.1).*

Generalized Newton methods of this kind were also considered by Kummer [21], Qi and Sun [27] and Qi [26]. In order for Algorithm 1 to be well-defined, we have to guarantee that there is always a nonsingular matrix $G_k \in \partial F(z^k)$ (under appropriate assumptions on the matrix M involved in the LCP(q, M)) and that a steplength $t_k > 0$ can always be found, i.e. that Δz^k is a suitable descent direction for the merit function $\|F\|^2$. An answer to the former problem will be given in Section 3, whereas the latter problem will shortly be discussed in Section 4. We stress, however, that Algorithm 1 is mainly used as a theoretical tool in this paper in order to prove finite termination of Harker and Pang's algorithm, see Section 4.

We note that in practical computations it seems preferable to use a nonmonotone line search instead of step (S.3). For instance, we have successfully used one modelled on the strategy by Grippo, Lampariello and Lucidi [10].

3 Finite Termination

In this section, we will prove that Algorithm 1 terminates in one step for linear complementarity problems provided that the current iterate $z^k := (x^k, y^k)$ is in a sufficiently small neighbourhood of a solution $z^* := (x^*, y^*)$ of the LCP(q, M). The following lemma is the key ingredient for this result.

Lemma 3 *Let* $z^* := (x^*, y^*) \in \mathbb{R}^{2n}$ *denote a solution of the LCP*(q, M)*. Then there is a positive number* $\epsilon(z^*)$ *such that*

$$F(z) - G(z - z^*) = 0 \tag{7}$$

for all $z \in B_{\epsilon(z^*)}$ and all matrices $G \in \partial F(z)$, where $B_\epsilon := \{z \in \mathbb{R}^{2n} \mid \|z - z^*\| < \epsilon\}$.

Proof. For an arbitrary $z = (x, y) \in \mathbb{R}^{2n}$, we obtain

$$Mx + q - y - (M(x - x^*) - (y - y^*)) = Mx^* + q - y^* = 0,$$

i.e., the first n equations of system (7) are satisfied. Now it will be shown that the remaining equations

$$\Delta(x_i, y_i) := \min\{x_i, y_i\} - a_i(x_i - x_i^*) - (1 - a_i)(y_i - y_i^*) = 0 \quad (i \in \mathcal{I}),$$

where a_i comes from the representation of the generalized Jacobian (cf. (3), (5)), are fulfilled in a certain neighbourhood of the solution $z^* = (x^*, y^*)$. If there is at least one index $i \in \mathcal{I}$ with $x_i^* + y_i^* > 0$, then set

$$\epsilon(z^*) := \frac{1}{3} \min\{x_i^* + y_i^* \mid i \in \mathcal{I}, x_i^* + y_i^* > 0\}.$$

Otherwise, let $\epsilon(z^*)$ be any positive number. Now, let $z \in B_{\epsilon(z^*)}$ and $G \in \partial F(z)$ be arbitrarily chosen. We distinguish the following cases:

(a) If $x_i^* = y_i^* = 0$ and $x_i = y_i$, we directly get $\Delta(x_i, y_i) = 0$.
(b) If $x_i^* = y_i^* = 0$ and $x_i < y_i$, it follows that $a_i = 1$ and therefore $\Delta(x_i, y_i) = 0$.
(c) If $x_i^* = y_i^* = 0$ and $x_i > y_i$, the same is obvious with $a_i = 0$.
(d) If $x_i^* = 0 < y_i^*$, we obtain $y_i \geq y_i^* - \epsilon(z^*) \geq 2\epsilon(z^*) > \epsilon(z^*) \geq |x_i - x_i^*| \geq x_i$. This yields $a_i = 1$ and $\Delta(x_i, y_i) = 0$.
(e) If $y_i^* = 0 < x_i^*$, it follows analogously to case (d) that $a_i = 0$ and $\Delta(x_i, y_i) = 0$.

Consequently, we have $\Delta(x_i, y_i) = 0$ for all $i \in \mathcal{I}$ and all $z \in B_{\epsilon(z^*)}$. □

We note that Lemma 3 is true even for infeasible vectors $z = (x, y) \in B_{\epsilon(z^*)}$, i.e., $y = Mx + q$ need not be satisfied.

Theorem 4 *Let $z^* := (x^*, y^*) \in \mathbb{R}^{2n}$ denote a solution of the LCP(q, M). If $z^k \in B_\epsilon$ for some sufficiently small $\epsilon > 0$ and if $G_k \in \partial F(z^k)$ is nonsingular, then z^{k+1} as generated by Algorithm 1 solves the LCP(q, M).*

Proof. Let $\epsilon := \epsilon(z^*)$ be as in Lemma 3. Based on this lemma and the nonsingularity of G_k, step (S.2) of Algorithm 1 yields

$$(z^k + \Delta z^k) - z^* = z^k - G_k^{-1} F(z^k) - z^* = -G_k^{-1}(F(z^k) - G_k(z^k - z^*)) = 0.$$

Therefore, we have

$$\|F(z^k + \Delta z^k)\|^2 = \|F(z^*)\|^2 = 0 \leq (1-\sigma)\|F(z^k)\|^2,$$

i.e., step (S.3) of Algorithm 1 computes $t_k = 1$ and step (S.4) provides $z^{k+1} = z^k + \Delta z^k = z^*$. $\square$

The proof of this theorem is based on an idea of Kojima and Shindo [20] in connection with Newton's method for piecewise continuously differentiable equations, see also Kummer [21] and Fischer [7] for related results.

Theorem 4 raises the following question - under what assumptions can we find a nonsingular matrix $G_k \in \partial F(z^k)$? The following results give sufficient conditions.

Theorem 5 *Let $M \in \mathbb{R}^{n \times n}$ be a P-matrix and let $z \in \mathbb{R}^{2n}$ be an arbitrary vector. Then every element $G \in \partial F(z)$ is nonsingular.*

Proof. Let $z \in \mathbb{R}^{2n}$ and $G \in \partial F(z)$ be arbitrary. Then, we consider $p = (p^{(1)}, p^{(2)}) \in \mathbb{R}^{2n}$ such that $Gp = 0$ is satisfied. With regard to (3) we have

$$Mp^{(1)} - p^{(2)} = 0, \tag{8}$$

$$a_i p_i^{(1)} + (1-a_i)p_i^{(2)} = 0 \quad (i \in \mathcal{I}), \tag{9}$$

where $a_i \in [0,1]$. From (8), we obtain $p^{(2)} = Mp^{(1)}$ and therefore $p_i^{(2)} = (Mp^{(1)})_i = M_{i\cdot}p^{(1)}$. Together with (9) it follows that

$$a_i p_i^{(1)} + (1-a_i)M_{i\cdot}p^{(1)} = 0 \quad (i \in \mathcal{I}). \tag{10}$$

Multiplying the ith equation by $p_i^{(1)}$ and recalling that M is a P-matrix, we immediately get $p^{(1)} = 0$. This and (8) yields $p^{(2)} = 0$. Consequently, the matrix G is indeed nonsingular. $\square$

As a direct consequence of Theorems 4 and 5, we get the result that Algorithm 1 possesses the finite termination property for P-matrix linear complementarity problems independent of the particular choice of the matrix $G \in \partial F(z)$.

The following theorem deals with the case where the matrix M is only assumed to be nondegenerate and states a somewhat weaker assertion.

Theorem 6 *Let $M \in \mathbb{R}^{n \times n}$ be a nondegenerate matrix and let $z \in \mathbb{R}^{2n}$ be an arbitrarily chosen vector. Then all elements $G \in \partial_B F(z)$ are nonsingular.*

Proof. The proof is similar to the one given for Theorem 3.1 in [16]. Let $z \in \mathbb{R}^n$ and $G \in \partial_B F(z)$ be arbitrary. According to (3) and (4) there is a vector $a \in \mathbb{R}^n$ with $a_i \in \{0,1\}$ such that $G = G(a)$. Define the index sets

$$J := \{i \in \mathcal{I} | a_i = 0\} \quad \text{and} \quad \bar{J} := \{i \in \mathcal{I} | a_i = 1\}.$$

Since $a_i \in \{0,1\}$, we have $\bar{J} = \mathcal{I} \setminus J$. Let $p = (p^{(1)}, p^{(2)}) \in \mathbb{R}^{2n}$ with $Gp = 0$, i.e., let equations (8) and (9) hold. The latter and the definitions of J and $\bar{J}$ yield

$$p^{(1)}_{\bar{J}} = 0 \quad \text{and} \quad p^{(2)}_{J} = 0. \tag{11}$$

Using this, it follows from (8) that

$$M_{JJ} p^{(1)}_J = 0. \tag{12}$$

Since M is nondegenerate, the submatrix M_{JJ} is nonsingular. Therefore, equation (12) implies

$$p^{(1)}_J = 0.$$

This together with the first equation in (11) yields $p^{(1)} = 0$ and we obtain from (8) that $p^{(2)} = 0$. Thus, $p = 0$. Therefore, all matrices $G \in \partial_B F(z)$ are nonsingular for any $z \in \mathbb{R}^n$. □

If $z \in \mathcal{D}$, then $\partial_B F(z)$ consists solely of the Jacobian matrix $\nabla F(z)$, which must then be nonsingular for nondegenerate matrices M because of Theorem 6. The following example, however, shows that if $z = (x, y)$ is a vector not belonging to $\mathcal{D}$, then singular matrices $G \in \partial F(z)$ might exist.

Example 7 *Let $n = 1, M = (-1) \in \mathbb{R}^{1\times 1}, q \in \mathbb{R}^1$ arbitrary and $z = (x, y) \in \mathbb{R}^2$ such that $x = y$. Obviously, M is a nondegenerate matrix, but not a P-matrix. From the representation of the generalized Jacobian $\partial F(z)$, it follows directly that the singular matrix*

$$G = \begin{pmatrix} -1 & -1 \\ 0.5 & 0.5 \end{pmatrix}$$

is an element of $\partial F(z)$ (just take $a = 0.5$).

Based on the following example, it is clear that it may happen that there is no nonsingular element $G \in \partial F(z)$ if the underlying matrix M is degenerate (i.e., not nondegenerate).

Example 8 *Let* $n = 1, M = (0) \in \mathbb{R}^{1\times 1}, q \in \mathbb{R}^1$ *arbitrary and* $z = (x, y) \in \mathbb{R}^2$ *such that* $x > y$*. Then* M *is degenerate and* $\partial F(z)$ *solely contains the singular matrix*

$$G = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}.$$

Now let $z^* := (x^*, y^*) \in \mathbb{R}^{2n}$ be a solution of LCP(q, M), and let α, β and γ denote the corresponding index sets as defined in Section 2. Without loss of generality we can partition the matrix M as well as the diagonal matrix D_a as follows:

$$M = \begin{pmatrix} M_{\alpha\alpha} & M_{\alpha\beta} & M_{\alpha\gamma} \\ M_{\beta\alpha} & M_{\beta\beta} & M_{\beta\gamma} \\ M_{\gamma\alpha} & M_{\gamma\beta} & M_{\gamma\gamma} \end{pmatrix}, \qquad D_a = \begin{pmatrix} D_{a,\alpha} & & \\ & D_{a,\beta} & \\ & & D_{a,\gamma} \end{pmatrix},$$

where $D_{a,\alpha} = (D_a)_{\alpha\alpha}$ etc. Similarly, for an arbitrary vector $q \in \mathbb{R}^n$, we use the corresponding partitioning

$$q = (q_\alpha, q_\beta, q_\gamma).$$

Based on this notation, we are able to formulate and prove our next results.

Theorem 9 *Let* $z^* := (x^*, y^*) \in \mathbb{R}^{2n}$ *be an R-regular solution of LCP*(q, M)*. Then all elements* $G \in \partial F(z^*)$ *are nonsingular.*

Proof. Taking an arbitrary $G \in \partial F(z^*)$, we consider $p = (p^{(1)}, p^{(2)}) \in \mathbb{R}^{2n}$ such that $Gp = 0$. With regard to (3) as well as to (8) and (9), we get

$$D_a p^{(1)} + (I - D_a) M p^{(1)} = 0 \tag{13}$$

and $p^{(2)} = Mp^{(1)}$. Therefore, we only have to show that $p^{(1)} = 0$ follows from (13). Using the definition of the index sets α, β, γ and formula (3) together with the representation (5) of $\partial F(z)$, it follows in the above notation that $D_{a,\alpha} = 0_{\alpha\alpha}$ and $D_{a,\gamma} = I_{\gamma\gamma}$. Consequently, (13) can be rewritten as

$$M_{\alpha\alpha} p^{(1)}_\alpha + M_{\alpha\beta} p^{(1)}_\beta + M_{\alpha\gamma} p^{(1)}_\gamma = 0_\alpha, \tag{14}$$

$$D_{a,\beta} p^{(1)}_\beta + (I - D_{a,\beta}) \left[M_{\beta\alpha} p^{(1)}_\alpha + M_{\beta\beta} p^{(1)}_\beta + M_{\beta\gamma} p^{(1)}_\gamma \right] = 0_\beta, \tag{15}$$

$$p^{(1)}_\gamma = 0_\gamma. \tag{16}$$

According to (16) and the R-regularity of z^*, we get

$$p_\alpha^{(1)} = -M_{\alpha\alpha}^{-1} M_{\alpha\beta} p_\beta^{(1)} \tag{17}$$

from (14). Substituting this into (15) and rearranging terms yields

$$\left(D_{a,\beta} + (I - D_{a,\beta})\left[M_{\beta\beta} - M_{\beta\alpha} M_{\alpha\alpha}^{-1} M_{\alpha\beta}\right]\right) p_\beta^{(1)} = 0_\beta. \tag{18}$$

Due to the R-regularity the bracketed term in (18) is a P-matrix. Hence, we can argue as in the proof of Theorem 5 that $p_\beta^{(1)} = 0_\beta$. This implies $p_\alpha^{(1)} = 0_\alpha$ (cf. (17)). Because of (16) the proof is complete. □

Theorem 10 *Let $z^* = (x^*, y^*) \in \mathbb{R}^{2n}$ be a b-regular solution of LCP(q, M). Then all elements $G \in \partial_B F(z^*)$ are nonsingular.*

Proof. Let $G \in \partial_B F(z^*)$ be arbitrarily chosen. Keeping the representation (4) of the B-subdifferential $\partial_B F(z)$ and formula (3) in mind, we see that an index set δ exists with $\alpha \subseteq \delta \subseteq \alpha \cup \beta$ and

$$a_\delta = 0_\delta, \; a_{\bar\delta} = e_{\bar\delta}, \tag{19}$$

where $\bar\delta = \mathcal{I} \setminus \delta$. If we write

$$M = \begin{pmatrix} M_{\delta\delta} & M_{\delta\bar\delta} \\ M_{\bar\delta\delta} & M_{\bar\delta\bar\delta} \end{pmatrix}, \qquad D_a = \begin{pmatrix} D_{a,\delta} & \\ & D_{a,\bar\delta} \end{pmatrix}$$

and assume that $Gp = 0$ for some vector $p = (p^{(1)}, p^{(2)}) \in \mathbb{R}^{2n}$, then (13) can be rewritten as

$$D_{a,\delta} p_\delta^{(1)} + (I_{\delta\delta} - D_{a,\delta})\left[M_{\delta\delta} p_\delta^{(1)} + M_{\delta\bar\delta} p_{\bar\delta}^{(1)}\right] = 0_\delta, \tag{20}$$

$$D_{a,\bar\delta} p_{\bar\delta}^{(1)} + (I_{\bar\delta\bar\delta} - D_{a,\bar\delta})\left[M_{\bar\delta\delta} p_\delta^{(1)} + M_{\bar\delta\bar\delta} p_{\bar\delta}^{(1)}\right] = 0_{\bar\delta}. \tag{21}$$

With regard to (3) and (19), we have $D_{a,\delta} = 0_{\delta\delta}$ and $D_{a,\bar\delta} = I_{\bar\delta\bar\delta}$. Therefore, (20) and (21) can be reduced to

$$p_{\bar\delta}^{(1)} = 0_{\bar\delta}, \; M_{\delta\delta} p_\delta^{(1)} = 0_\delta.$$

Owing to the b-regularity assumption, the submatrix $M_{\delta\delta}$ is nonsingular and we get

$$p_\delta^{(1)} = 0_\delta.$$

This shows that $p^{(1)} = 0$. Finally, $p^{(2)} = Mp^{(1)} = 0$, according to (8), implies $p = 0$. $\square$

From well-known properties of the B-subdifferential and the generalized Jacobian, see [26,2], it follows that the nonsingularity results given in Theorems 9 and 10 remain true in a certain small neighbourhood of the solution z^*. Hence we can also apply Theorem 4 under the R- and b-regularity assumptions. In contrast to the case of a nondegenerate or P-matrix complementarity problem, however, the neighbourhood for the finite termination is possibly smaller and not necessarily given by the constant $\varepsilon(z^*)$ defined in the proof of Lemma 3.

In order to illustrate the finite termination of Algorithm 1, we implemented that algorithm in MATLAB (using a nonmonotone line search) and tested it on the following problem found in Harker and Pang [12].

Example 11 *Let*

$$M := A^T A + B + diag(c_i).$$

The matrix $A \in \mathrm{I\!R}^{n\times n}$, the skew-symmetric matrix $B \in \mathrm{I\!R}^{n\times n}$ and the vectors $c, q \in \mathrm{I\!R}^n$ are randomly generated with uniformly distributed entries

$$a_{ij},\, b_{ij} \in (-5, 5), \quad c_i \in (0, 0.3), \quad q_i \in (-500, 0) \quad (i, j \in \mathcal{I}).$$

In our MATLAB-implementation we chose $G_k \in \partial F(z^k)$ such that $a_i = 1$ if $x_i^k < y_i^k$ and otherwise $a_i = 0$. Figure 1 shows the convergence history of Algorithm 1 for one typical instance of Example 11. This figure is interesting since it clearly demonstrates both the nonmonotone line search and the finite termination (note that the norm $\|F(z^k)\|/\|F(z^0)\|$ for the sixth iterate $k = 6$ is relatively large, whereas $\|F(z^7)\|/\|F(z^0)\|$ is almost around the machine precision).

4 Application to Harker and Pang's Algorithm

We will first give a review of Harker and Pang's algorithm [12], see also Pang [24]. They have considered the characterization $H(x) = 0$ of the LCP(q, M)

with $H : \mathbb{R}^n \to \mathbb{R}^n$ being defined by

$$H(x) = \min\{x, Mx + q\}. \tag{22}$$

Their algorithm is as follows:

Algorithm 2 *(Harker-Pang method)*

(S.0) *Choose* $x^0 \in \mathbb{R}^n, \beta, \sigma \in (0,1)$ *and set* $k := 0$.
(S.1) *If* $\|H(x^k)\| = 0$, *stop. (x^k solves the LCP(q, M).)*
(S.2) *Compute* Δx^k *as a solution of the generalized Newton equation*

$$BH(x^k; \Delta x) = -H(x^k). \tag{23}$$

(S.3) *Set* $t_k := \beta^{m_k}$ *with* m_k *being the smallest nonnegative integer* m *satisfying the Armijo condition*

$$\|H(x^k + \beta^m \Delta x^k)\|^2 \le (1 - \beta^m \sigma)\|H(x^k)\|^2.$$

(S.4) *Set* $x^{k+1} := x^k + t_k \Delta x^k,\ k := k + 1$ *and go to(S.1).*

By $BH(x; \cdot)$ the B-derivative of H at x is denoted, see e.g. [13,24] for details. In particular, $BH(x; \cdot)$ is a positive homogeneous but not necessarily linear mapping. Therefore, in contrast to Algorithm 1, the Newton equation (23) of Algorithm 2 is in general nonlinear. For the particular function H, the B-derivative at x^k is given by

$$(BH(x^k; \Delta x))_i = \begin{cases} M_{i\cdot}\Delta x & \text{if } i \in \alpha(x^k) := \{i \in \mathcal{I} |\ (Mx^k + q)_i < x_i^k\} \\ \min\{M_{i\cdot}\Delta x, \Delta x_i\} & \text{if } i \in \beta(x^k) := \{i \in \mathcal{I} |\ (Mx^k + q)_i = x_i^k\} \\ \Delta x_i & \text{if } i \in \gamma(x^k) := \{i \in \mathcal{I} |\ (Mx^k + q)_i > x_i^k\} \end{cases}.$$

The generalized Newton equation (23) (with $\Delta x = \Delta x^k$) can therefore be rewritten as follows:

$$M_{i\cdot}\Delta x^k = -(Mx^k + q)_i \qquad (i \in \alpha(x^k)), \tag{24}$$

$$\min\{M_{i\cdot}\Delta x^k, \Delta x_i^k\} = -x_i^k \qquad (i \in \beta(x^k)), \tag{25}$$

$$\Delta x_i^k = -x_i^k \qquad (i \in \gamma(x^k)). \tag{26}$$

In order to avoid nonlinear subproblems, Harker and Pang proposed a modification of step (S.3) of Algorithm 2, see also [3]. This modification, together with a nondegenerate starting point x^0, ensures that all iterates x^k remain nondegenerate, i.e., $\beta(x^k) = \emptyset$ for $k = 0, 1, \ldots$ Consequently, the subproblems only consist of the linear equations (24) and (26).

A further modification of Algorithm 2 is obtained if the Newton-type algorithm given in [25] is used for linear complementarity problems LCP(q, M). There, roughly speaking, another index set definition possibly enlarges the dimension of the linear complementarity part (25) in the subproblems. However, stronger results on global and locally quadratic convergence can be proved.

It can easily be seen that the subsequent considerations, in particular Theorems 12 and 13, also hold for these modified versions of Harker and Pang's algorithm.

Throughout this section, we will follow Harker and Pang [12] and assume that the system (24)–(26) has a solution Δx^k for all iteration steps $k = 0, 1, \dots$ We will now show that (under this assumption) both Algorithm 1 and Algorithm 2 generate the same sequence $x^0, x^1, \dots$ if we take $y^0 := Mx^0 + q$ in Algorithm 1 and choose a particular matrix $G_k \in \partial F(z^k)$. From $y^0 = Mx^0 + q$ and a simple induction argument, we first get

$$y^k = Mx^k + q \quad (k = 0, 1, \dots). \tag{27}$$

Let $\Delta x^k \in \mathbb{R}^n$ denote a solution of (24)–(26) and define

$$a_i := a_i^k := \begin{cases} 1 & \text{if } \Delta x_i^k \leq M_{i\cdot} \Delta x^k \\ 0 & \text{if } \Delta x_i^k > M_{i\cdot} \Delta x^k \end{cases} \qquad (i \in \beta(x^k)). \tag{28}$$

From this choice of a_i it follows that the corresponding matrix G_k is an element of the B-subdifferential $\partial_B F(z^k)$. Moreover, using (28) and (27), it can easily be verified that the system $G_k \Delta z^k = -F(z^k)$ leads to exactly the same system as (24)–(26), i.e., both algorithms are indeed identical. Since Algorithm 1 has the finite termination property for nondegenerate matrices M with the choice of a_i given in (28), it follows directly from the results of Section 3 that the same holds for Algorithm 2. We will summarize these considerations in the following theorem, which answers the unsolved question raised by Harker and Pang [12].

Theorem 12 *Let $M \in \mathbb{R}^{n \times n}$ be a nondegenerate matrix, and let x^* be a solution of the LCP(q, M). If $x^k \in B_\epsilon$ for some small enough $\epsilon > 0$, and if the generalized Newton-equation (23) has a solution Δx^k, then the next iterate x^{k+1} generated by Algorithm 2 will coincide with x^*.*

As noted in [12], the system (23) is always solvable for P-matrix linear complementarity problems. Therefore, if we assume $M \in \mathbb{R}^{n \times n}$ being a P-matrix, the additional assumption that the system (23) is solvable can be omitted in Theorem 12.

From Theorem 10, we further get the following result.

Theorem 13 *Let x^* be a b-regular solution of LCP(q, M). If $x^k \in B_\epsilon$ for some small enough $\epsilon > 0$, and if the generalized Newton-equation (23) has a solution Δx^k, then the next iterate x^{k+1} generated by Algorithm 2 will coincide with x^*.*

Once again, we note that if the b-regularity assumption of Theorem 13 is replaced by the stronger R-regularity, the additional assumption of the solvability of the subproblems (23) becomes superfluous.

We conclude this section by noting that Harker and Pang [12] were able to prove that their search direction Δx^k is a descent direction for $\|H\|^2$. This in turn implies that the search direction Δz^k obtained in Algorithm 1 for the choice of $G_k \in \partial F(z^k)$ based on (28), is also a descent direction for $\|F\|^2$. However, this particular choice of G_k is unknown a priori.

5 Final Remarks

In this paper, we proved that a certain iterative method for the solution of linear complementarity problems terminates after a finite number of steps under suitable assumptions. In particular, this result can be applied to an algorithm of Harker and Pang [12] and gives an affirmative answer to an unsolved question raised by these authors.

In order to prove finite termination of Harker and Pang's method, we introduced (as a theoretical tool) the generalized Newton method described in Algorithm 1. Here we want to mention that there is a close relationship between Algorithm 1 and (block) principal pivoting methods (see, e.g., Júdice [14] for a recent survey). The central idea of pivoting methods is to maintain the equations

$$y^k = Mx^k + q, \quad x_i^k y_i^k = 0 \quad (i \in \mathcal{I}) \tag{29}$$

at all steps $k = 0, 1, \dots$ and to reach the feasibility condition $x^k \geq 0, y^k \geq 0$ during the iteration. Now consider Algorithm 1 and assume that the matrix G_k is always taken from the B-subdifferential $\partial_B F(z^k)$. Let $z^0 = (x^0, y^0)$ be a starting vector such that (29) is satisfied for $k = 0$ (e.g., take $x^0 = 0, y^0 = q$). Then it is not difficult to see that (29) holds for all $k = 0, 1, \dots$ as long as the full stepsize $t_k = 1$ is accepted by Algorithm 1. Although the line search procedure in Step (S.3) distinguishes Algorithm 1 from pivoting methods this procedure can be viewed as another kind of pivoting rule, see also [12,3].

Acknowledgement

We wish to thank the referees, the associate editor and Professor J. J. Júdice for their helpful comments.

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