A Continuation Method for the Solution of Monotone Variational Inequality Problems

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Abstract

We describe a new continuation method for the solution of monotone variational inequality problems. The method follows a smooth path which is formed by the solutions of certain perturbed problems. We give some conditions for the existence of this path and present numerical results for variational inequality, convex optimization and complementarity problems.

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Abbreviated title. Monotone Variational Inequalities.

1 Introduction

Consider the *variational inequality problem*, denoted by VIP(X, F), which is to find a vector $x^* \in X$ such that

$$F(x^*)^T(x - x^*) \geq 0 \text{ for all } x \in X,$$

where $F : \Re^n \to \Re^n$ is a given function and X is a nonempty closed subset of $\Re^n$.

This problem has several important applications, e.g., in several equilibrium models arising in economics, transportation and engineering sciences, see [5, 12] for some examples. The variational inequality problem also covers some other well-known mathematical problems. For example, consider the nonlinear complementarity problem, NCP(F) for short: Find a vector in $\Re^n$ satisfying the conditions

$$x \geq 0, F(x) \geq 0, x^T F(x) = 0.$$

It is not difficult to see that VIP(X, F) reduces to NCP(F) if $X = \Re^n_+$. Another instance of VIP(X, F) is the constrained optimization problem

$$\min f(x) \text{ s.t. } x \in X,$$

where $f : \Re^n \to \Re$. If both the objective function f and the feasible set X are convex, it follows immediately from the optimality conditions in convex programming that this constrained optimization problem is equivalent to VIP$(X, \nabla f)$.

There exist several solution methods for solving VIP(X, F), the interested reader is referred to the survey papers [10, 23]. Among the most successful methods are the (globalized) linearization methods ([8, 17, 18, 29, 36]), the nonsmooth equation-based methods ([21, 22, 34, 35]) and the continuation methods ([2, 16, 31]). The method to be described in this paper belongs to the class of continuation methods. It can be viewed as a method which lies between the method considered by Chen and Harker [2] and the method introduced by the authors in [16]. Chen and Harker's method is able to handle monotone problems, but depends on four perturbation parameters, whereas the method described in [16] is only able to handle strongly monotone variational inequality problems, but, on the other hand, depends on just one perturbation parameter. Note that these parameters play a crucial role in a successful implementation of these algorithms, and that it is in general difficult to find "optimal" updating rules for these parameters. For this reason it is usually advisable to use as few parameters as possible.

The method considered here depends on two parameters, one is the barrier parameter known from interior-point methods, the other one is a regularization parameter which allows us to extend the approach given in [16] for strongly monotone variational inequalities to monotone problems.

The paper is organized as follows: In Section 2, we give a short review of some basic concepts which will be used in the subsequent sections. The continuation method is described in Section 3, which also contains some convergence results for this method. In Section 4 we prove a local error bound result showing that the solutions of our perturbed problems can be made arbitrarily close to the solution of the original problem. In Section 5 we present some numerical results for our method applied to several variational inequality, constrained optimization and complementarity problems taken from the literature. We conclude with some final remarks in Section 6.

2 Background Material

In this section, we summarize some preliminary facts which will be useful subsequently.

2.1 Definition. Let $F : X \to \Re^n$ be a function. F is said to be *strongly monotone* on the set X with modulus $\alpha > 0$ if

$$(x - y)^T (F(x) - F(y)) \geq \alpha \|x - y\|^2 \quad \forall x, y \in X.$$

F is said to be *monotone* on X if the above inequality holds when α is replaced by 0.

It is well known [10] that, when F is continuously differentiable, F is strongly monotone on X with modulus α if and only if

$$d^T \nabla F(x) d \geq \alpha \|d\|^2 \quad \text{for all } x \in X, d \in \Re^n,$$

and F is monotone on X if and only if

$$d^T \nabla F(x) d \geq 0 \quad \text{for all } x \in X, d \in \Re^n.$$

We also note that VIP(X, F) has a unique solution if F is strongly monotone on X, see [10].

In most applications, the set X has a simple structure and can be described as

$$X = \{x \in \Re^n \,|\, g(x) \geq 0, h(x) = 0\},$$

where $g : \Re^n \to \Re^m$ and $h : \Re^n \to \Re^p$ are twice continuously differentiable functions such that the component functions $g_i : \Re^n \to \Re$ $(i \in I := \{1, \dots, m\})$ are concave and the component functions $h_j : \Re^n \to \Re$ $(j \in J := \{1, \dots, p\})$ are affine-linear. Note that this implies that the feasible set X is convex.

Throughout this paper, we assume that X is defined in this way. More precisely, we assume for simplicity that $J = \emptyset$ since our entire analysis goes through for this more general case in a straightforward way.

Associated with the above structure of the constraint set X, there is a well known constraint qualification which is defined as follows.

2.2 DEFINITION. A vector $x \in X$ satisfies the *linear independence constraint qualification*, LICQ for short, if the gradients $\nabla g_i(x)$ $(i \in I(x) := \{i \in I| \; g_i(x) = 0\}$ of the active inequality constraints are linearly independent.

Suppose x^* is a solution of VIP(X, F). If the LICQ holds at x^*, then there exist vectors $y^* \in \Re^m$ and $z^* \in \Re^m$ such that the triple $w^* := (x^*, y^*, z^*)$ satisfies the following KKT-conditions,

$$\begin{aligned} F(x^*) - \nabla g(x^*)^T y^* &= 0, \\ g(x^*) - z^* &= 0, \\ y^* \geq 0, z^* \geq 0, y_i^* z_i^* &= 0 \quad (i \in I). \end{aligned}$$

By the concavity of g_i $(i \in I)$, the converse is also true, namely, the x-part of any KKT-triple must be a solution of VIP(X, F), see [10]. Thus, under the LICQ assumption, the KKT-conditions and problem VIP(X, F) itself are completely equivalent. The following second order condition will be used in Section 3.

2.3 DEFINITION. Let $L(x, y) = F(x) - \nabla g(x)^T y$ and let $w^* = (x^*, y^*, z^*)$ be a KKT-triple of VIP(X, F). The *strong second order sufficient condition*, SSOSC for short, is satisfied at w^* if

$$d^T \nabla_x L(x^*, y^*) d > 0 \quad \text{for all } d \in \mathcal{T}(x^*), d \neq 0,$$

where

$$\mathcal{T}(x^*) = \{d \in \Re^n | \nabla g_i(x^*) = 0, i \in I^* := I(x^*)\}.$$

Now let $G : \Re^n \to \Re^n$ be an arbitrary function which is locally Lipschitz–continuous at a vector $x \in \Re^n$. Then the generalized Jacobian of G at x is defined by

$$\partial G(x) := \text{conv}\{H \in \Re^{n \times n} | \, \exists \{x^k\} \subseteq D_G : x^k \to x, \nabla G(x^k) \to H\},$$

where D_G denotes the set of differentiable points of G and conv$(\mathcal{A})$ is the convex hull of a set $\mathcal{A}$. It is known that $\partial G(x)$ is a nonempty, convex and compact set, see [4]. Based on the notion of the generalized Jacobian, we can define the class of semismooth functions.

2.4 DEFINITION. Let $G : \Re^n \to \Re^n$ be locally Lipschitzian at $x \in \Re^n$. We say that G is *semismooth* at x if

$$\lim_{\substack{H \in \partial G(x + tv') \\ v' \to v, t \downarrow 0}} Hv'$$

exists for any $v \in \Re^n$.

Semismooth functionals were introduced in [20] and extended to vector-valued mappings in [27]. It is known that a mapping $G : \Re^n \to \Re^n$ is semismooth if and only if all its component functions $G_i : \Re^n \to \Re$ are semismooth. Smooth, piecewise smooth and convex functionals are known to be semismooth. The composite of two semismooth functions is again semismooth. The reader is referred to [20, 27, 26] for some further properties and more comprehensive discussions of semismooth functions.

3 Continuation Method

The continuation method to be presented next is based on the following perturbation of the KKT-conditions:

$$\begin{aligned} F(x) + \varepsilon x - \nabla g(x)^T y &= 0, \\ g(x) - z &= 0, \\ y \geq 0, z \geq 0, y_i z_i &= \mu \quad (i \in I); \end{aligned}$$

here, the real numbers $\varepsilon \geq 0$ and $\mu \geq 0$ are the perturbation parameters. We call this perturbed system the perturbed variational inequality problem and denote it by PVIP(X, F, μ, ε).

Given any fixed parameter $\mu \geq 0$, let us define a functional $\varphi_\mu : \Re^2 \to \Re$ by

$$\varphi_\mu(a, b) := a + b - \sqrt{(a-b)^2 + 4\mu}. \tag{1}$$

This function has recently been introduced in [15]. Its most interesting property is the following characterization of its zeros, see [15].

3.1 Lemma. *For every $\mu \geq 0$, we have*

$$\varphi_\mu(a, b) = 0 \Longleftrightarrow a \geq 0, b \geq 0, ab = \mu.$$

Let the operator $\Phi : \Re^n \times \Re^m \times \Re^m \to \Re^n \times \Re^m \times \Re^m$ be defined by

$$\Phi(w; \mu, \varepsilon) := \Phi(x, y, z; \mu, \varepsilon) := \begin{pmatrix} F(x) + \varepsilon x - \nabla g(x)^T y \\ g(x) - z \\ \varphi_\mu(y, z) \end{pmatrix},$$

where

$$\varphi_\mu(y, z) := (\varphi_\mu(y_1, z_1), \ldots, \varphi_\mu(y_m, z_m))^T \in \Re^m.$$

Using Lemma 3.1, it is straightforward to verify the following result.

3.2 Theorem. *A vector $w(\mu, \varepsilon) := (x(\mu, \varepsilon), y(\mu, \varepsilon), z(\mu, \varepsilon)) \in \Re^n \times \Re^m \times \Re^m$ is a solution of the perturbed variational inequality problem PVIP(X, F, μ, ε) if and only if $w(\mu, \varepsilon)$ solves the nonlinear system of equations $\Phi(w; \mu, \varepsilon) = 0$.*

Theorem 3.2 serves as motivation for the following continuation method.

3.3 Algorithm.

(S.0) Let $\{\mu_k\}, \{\varepsilon_k\} \subseteq \Re$ be two decreasing sequences with $\lim_{k\to\infty} \mu_k = 0$ and $\lim_{k\to\infty} \varepsilon_k = 0$. Choose $w^0 := (x^0, y^0, z^0) \in \Re^n \times \Re^m \times \Re^m$, and set $k := 0$.

(S.1) If $\|\Phi(w^k; 0, 0)\| = 0$, stop: w^k is a KKT-point of VIP(X, F).

(S.2) Find a solution $w^{k+1} := w(\mu_{k+1}, \varepsilon_{k+1})$ of the nonlinear system of equations

$$\Phi(w; \mu_{k+1}, \varepsilon_{k+1}) = 0$$

(or, equivalently, of the perturbed problem PVIP$(X, F, \mu_{k+1}, \varepsilon_{k+1})$).

(S.3) Set $k := k + 1$, and go to (S.1).

In the remaining part of this section, we investigate two different conditions for ensuring the existence, boundedness and convergence of the sequence $\{w(\mu_k, \varepsilon_k)\}$ as generated by Algorithm 3.3.

We first show that if LICQ and SSOSC are satisfied at a KKT-triple $w^* := (x^*, y^*, z^*)$ of VIP(X, F), then $w(\mu, \varepsilon)$ exists uniquely for all $\mu > 0$ and all sufficiently small $\varepsilon > 0$, and that the whole sequence $\{w(\mu_k, \varepsilon_k)\}$ converges to w^* as μ_k and ε_k converge to 0 (in an arbitrary way).

In the second type of conditions, we suppose that LICQ holds at every feasible point $x \in X$, and that $\mu_k \to 0$ and $\varepsilon_k \to 0$ in such a way that the sequence $\{\mu_k/\varepsilon_k\}$ remains bounded. Under these assumptions we prove that the perturbed problems PVIP$(X, F, \mu_k, \varepsilon_k)$ have a unique solution $w(\mu_k, \varepsilon_k)$ for all $\mu_k > 0$ and all $\varepsilon_k > 0$, that every accumulation point of the sequence $\{w(\mu_k, \varepsilon_k)\}$ is a KKT-triple of VIP(X, F) and that there exists at least one such accumulation point.

To begin, we first investigate the operator $\Phi(.; 0, 0)$. Note that the function φ_μ is not everywhere differentiable in the limiting case $\mu = 0$. Hence $\Phi(.; 0, 0)$ is also not everywhere differentiable. However, the following result holds.

3.4 Lemma. *The function $\Phi(.; 0, 0)$ is semismooth.*

Proof. Recall that $\Phi(.; 0, 0)$ is semismooth if and only if all component functions are semismooth. Since the first $n + m$ component functions of $\Phi(.; 0, 0)$ are continuously differentiable, it suffices to consider the last m components. Since

$$\varphi_0(y_i, z_i) = y_i + z_i - \sqrt{(y_i - z_i)^2} = y_i + z_i - |y_i - z_i| = 2\min\{y_i, z_i\},$$

and since the min-function is piecewise smooth, all component functions of $\Phi(.; 0, 0)$ are semismooth. This proves the statement. □

3.5 Theorem. *Let $x^* \in \Re^n$ be a solution of VIP(X, F) and assume that LICQ and SSOSC are satisfied. Let $w^* := (x^*, y^*, z^*) \in \Re^n \times \Re^m \times \Re^m$ be the corresponding* *(unique) KKT-point of VIP(X, F). Then all $G \in \partial\Phi(w^*; 0, 0)$ are nonsingular.*

Proof. The proof is a slight variation of the one of Lemma 3.11 in [16]. The details are therefore omitted here. □

3.6 Corollary. *Let the assumptions of Theorem 3.5 be satisfied. Then w^* is the unique KKT-point of VIP(X, F) and x^* is the unique solution of VIP(X, F).*

Proof. By Lemma 3.4, Theorem 3.5 and Proposition 3 in [24], $w^* := (x^*, y^*, z^*)$ is a locally unique solution of the nonlinear equation $\Phi(w; 0, 0) = 0$ and therefore a locally unique KKT-point of VIP(X, F). Assume there exists a second solution $\bar{w} = (\bar{x}, \bar{y}, \bar{z})$ of $\Phi(w; 0, 0) = 0$. Since F is monotone, the solution set of VIP(X, F) is convex. Hence we must have $\bar{x} = x^*$ by the local uniqueness of w^*. But then we also have $\bar{z} = g(\bar{x}) = g(x^*) = z^*$. Moreover, from

$$F(x^*) - \nabla g(x^*)^T y^* = 0,$$

$$F(x^*) - \nabla g(x^*)^T \bar{y} = F(\bar{x}) - \nabla g(\bar{x})^T \bar{y} = 0,$$

$$y_i^* g_i(x^*) = 0, \bar{y}_i g_i(x^*) = \bar{y}_i g_i(\bar{x}) = 0$$

and the LICQ assumption, we also get $\bar{y} = y^*$. This proves the desired result. □

3.7 Corollary. *Let the assumptions of Theorem 3.5 be satisfied. Then there exists an $\bar{\varepsilon} > 0$ such that PVIP(X, F, μ, ε) has a unique solution $w(\mu, \varepsilon) = (x(\mu, \varepsilon), y(\mu, \varepsilon), z(\mu, \varepsilon)) \in \Re^n \times \Re^m \times \Re^m$ for all $0 < \varepsilon \leq \bar{\varepsilon}$ and all $\mu > 0$.*

Proof. In view of Theorem 3.5, it follows from Clarke's Implicit Function Theorem [4] that there exist $\bar{\varepsilon} > 0$ and $\bar{\mu} > 0$ such that PVIP(X, F, μ, ε) has a unique solution $w(\mu, \varepsilon)$ for all $0 < \varepsilon \leq \bar{\varepsilon}$ and all $0 < \mu \leq \bar{\mu}$. Since the set of active inequality constraints at a point x sufficiently close to x^* is a subset of the active inequality constraints at x^*, the LICQ assumption is also satisfied in a small neighbourhood of x^*. Hence, we get from Theorem 3.12 in [16] and the simple observation that the function $F_\varepsilon : \Re^n \to \Re^n$ defined by $F_\varepsilon(x) := F(x) + \varepsilon x$ is uniformly monotone that PVIP(X, F, μ, ε) has a unique solution $w(\mu, \varepsilon)$ for all $0 < \varepsilon \leq \bar{\varepsilon}$ and all $\mu > 0$. □

3.8 Corollary. *Let the assumptions of Theorem 3.5 be satisfied. Let $\{\mu\}$ and $\{\varepsilon\}$ be two sequences of positive numbers converging to 0, and let us denote the unique solution of the corresponding perturbed problem PVIP(X, F, μ, ε) by $w(\mu, \varepsilon)$. Then the entire sequence $\{w(\mu, \varepsilon)\}$ converges to the (unique) KKT-point w^* of VIP(X, F).*

Proof. We first note that the sequence $\{w(\mu, \varepsilon)\}$ exists for all $\mu > 0$ and all $\varepsilon > 0$ sufficiently small by Corollary 3.7. Next we note that it is an easy consequence of Theorem 3.5 and Clarke's Implicit Function Theorem [4] that the sequence $\{w(\mu, \varepsilon)\}$ remains bounded. Hence this sequence has at least one accumulation point $w(0, 0)$. Using simple continuity arguments, one can easily deduce that $w(0, 0)$ is a KKT-point of VIP(X, F). Hence we have shown that every accumulation point of the sequence $\{w(\mu,\varepsilon)\}$ is a KKT-point of VIP(X,F). By Corollary 3.6, however, VIP(X,F) has a unique KKT-point w^*, so the entire sequence $\{w(\mu,\varepsilon)\}$ must converge to w^*. □

We now come to the second set of sufficient conditions for the existence and the boundedness of the sequence $\{w(\mu_k,\varepsilon_k)\}$ and the convergence of the algorithm.

3.9 Lemma. *Suppose the LICQ holds at any point of X. Then PVIP(X,F,μ,ε) has a unique solution $w(\mu,\varepsilon)=(x(\mu,\varepsilon),y(\mu,\varepsilon),z(\mu,\varepsilon))\in\Re^n\times\Re^m\times\Re^m$ for all $\varepsilon>0$ and all $\mu>0$.*

Proof. For any fixed $\varepsilon>0$, VIP(X,F_ε) has a unique solution $x(\varepsilon)$ by the strong monotonicity of F_ε and the convexity of X, where $F_\varepsilon(x):=F(x)+\varepsilon x$. Since the LICQ holds at $x(\varepsilon)\in X$, the perturbed variational inequality problems PVIP(X,F,μ,ε) have a unique solution for all $\mu>0$ by Theorem 3.12 in [16]. Therefore, the desired result follows by taking into account that $\varepsilon>0$ is arbitrary. □

The following is a technical lemma which will be used in the proof of the main convergence theorem, Theorem 3.11 below.

3.10 Lemma. *Suppose $w^*=(x^*,y^*,z^*)$ is a KKT-point of VIP(X,F), and $w(\mu,\varepsilon)$ is a solution of PVIP(X,F,μ,ε) for some $\mu>0$ and $\varepsilon>0$. Then*

$$\varepsilon\|x(\mu,\varepsilon)-\frac{x^*}{2}\|^2\le\frac{\varepsilon}{4}\|x^*\|^2+m\mu,$$

where $w(\mu,\varepsilon)=(x(\mu,\varepsilon),y(\mu,\varepsilon),z(\mu,\varepsilon))$.

Proof. First note that $w(\mu,\varepsilon)$ is uniquely defined by Lemma 3.9. For simplicity, let w denote $w(\mu,\varepsilon)$. Since w is a solution of PVIP(X,F,μ,ε) and w^* is a KKT-point of VIP(X,F), it follows that

$$\begin{gathered}(F(x)+\varepsilon x)^T(x-x^*)-(\nabla g(x)(x-x^*))^Ty=0,\\ g(x)-z=0,\\ y>0,z>0,y_iz_i=\mu\quad(i\in I)\end{gathered}$$

and

$$\begin{gathered}F(x^*)^T(x^*-x)-(\nabla g(x^*)(x^*-x))^Ty^*=0,\\ g(x^*)-z^*=0,\\ y^*\ge0,z^*\ge0,y_i^*z_i^*=0\quad(i\in I).\end{gathered}$$

The addition of the first and fourth equations yields

$$\varepsilon x^T(x-x^*)+(F(x)-F(x^*))^T(x-x^*)=(\nabla g(x)(x-x^*))^Ty+(\nabla g(x^*)(x^*-x))^Ty^*. \quad (2)$$

In view of the monotonicity of F, the concavity of g_i $(i\in I)$ and the nonnegativeness of y and y^*, we have

$$(F(x) - F(x^*))^T(x - x^*) \geq 0,$$
$$(\nabla g(x)(x - x^*))^T y \leq (g(x) - g(x^*))^T y,$$
$$(\nabla g(x^*)(x^* - x))^T y^* \leq (g(x^*) - g(x))^T y^*.$$

It follows from (2) and the above three inequalities that

$$\varepsilon x^T(x - x^*) \leq (g(x) - g(x^*))^T y + (g(x^*) - g(x))^T y^*,$$

which is equivalent to

$$\varepsilon ||x - \frac{x^*}{2}||^2 \leq \frac{\varepsilon}{4}||x^*||^2 + (z - z^*)^T(y - y^*).$$

Since y, y^*, z and z^* are nonnegative, and $y_i^* z_i^* = 0$ and $y_i z_i = \mu$ ($i \in I$), the assertion follows from the above inequality. □

3.11 Theorem. *Suppose VIP(X, F) has a solution x^*. Suppose the LICQ holds at any point of X. Let $\{\mu_k\}$ and $\{\varepsilon_k\}$ be two sequences converging to* 0*. Then the algorithm is well-defined and PVIP($X, F, \mu_k, \varepsilon_k$) has a unique solution $w(\mu_k, \varepsilon_k)$ for any $\mu_k > 0$ and $\varepsilon_k > 0$. Furthermore, if there exists a positive constant α such that $\mu_k \leq \alpha\varepsilon_k$, then any accumulation point of the sequence $\{w(\mu_k, \varepsilon_k)\}$ is a KKT-point of VIP(X, F), and there exists at least one accumulation point.*

Proof. The existence and uniqueness of $w(\mu_k, \varepsilon_k)$ follows from Lemma 3.9. The second part of the theorem follows from the continuity of PVIP(X, F, μ, ε) with respect to w, μ and ε. Hence we only have to prove the existence of at least one accumulation point. We prove that the sequence $\{w^k\} := \{w(\mu_k, \varepsilon_k)\}$ remains bounded. The boundedness of the x-part follows immediately from Lemma 3.10 and the fact that $\mu_k = O(\varepsilon_k)$ by assumption. The boundedness of the z-part is therefore a consequence of the continuity of g and the equation $g(x^k) = z^k$. Assume that the y-part is unbounded. Without loss of generality, we can assume, subsequencing if necessary, that $y^k/\|y^k\| \to y^{**}$ for some y^{**} such that $\|y^{**}\| = 1$ and $x^k \to x^{**}, z^k \to z^{**}$. Let I^{**} denote the active index set at x^{**}, i.e., $I^{**} = \{i \in I |\, g_i(x^{**}) = 0\}$. Clearly, for all $i \notin I^{**}$, the sequences $\{y_i^k\}$ are bounded since $z_i^k = g_i(x^k)$ and $y_i^k z_i^k = \mu_k$. Therefore we have $y_i^{**} = 0$ for all $i \notin I^{**}$. Hence, dividing $F(x^k) + \varepsilon x^k - \nabla g(x^k)^T y^k = 0$ by $\|y^k\|$ and taking the limit, it follows

$$\sum_{i \in I^{**}} y_i^{**} \nabla g_i(x^{**}) = 0.$$

By the LICQ assumption, we therefore obtain $y_i^{**} = 0$ for all $i \in I^{**}$. Thus we have $y^{**} = 0$ which contradicts the fact that $\|y^{**}\| = 1$. This shows that the sequence $\{y^k\}$ is also bounded. □

The proof of the following result is very similar to the one of Theorem 3.5 in [16]. The proof is therefore omitted here.

3.12 THEOREM. *If $F : \Re^n \to \Re^n$ is monotone on X, then the operator $\Phi(.;\mu,\varepsilon)$ is continuously differentiable and the Jacobian matrices $\nabla\Phi(w;\mu,\varepsilon)$ are nonsingular for all $\mu > 0$, $\varepsilon > 0$ and all $w = (x, y, z) \in \Re^n \times \Re^m_+ \times \Re^m$. If, in addition, the functions g_i $(i \in I)$ are affine–linear, then the Jacobians $\nabla\Phi(w;\mu,\varepsilon)$ are nonsingular for all $\mu > 0$, $\varepsilon > 0$ and all $w = (x, y, z) \in \Re^n \times \Re^m \times \Re^m$.*

3.13 COROLLARY. *Under the assumptions of Theorem 3.12, the solution operator $(\mu,\varepsilon) \to w(\mu,\varepsilon)$ is continuously differentiable for all $\mu > 0$ and all $\varepsilon > 0$.*

PROOF. This follows from Theorem 3.12 and the Implicit Function Theorem, taking into account the fact that $y(\mu,\varepsilon) \geq 0$ for any solution $w(\mu,\varepsilon) = (x(\mu,\varepsilon), y(\mu,\varepsilon), z(\mu,\varepsilon))$ of PVIP(X, F, μ, ε). □

4 A Local Error Bound Result

In this section, we want to give a more precise measure of how close a solution $w(\mu,\varepsilon)$ of PVIP(X, F, μ, ε) is to a KKT-point w^* of the original problem VIP(X, F). To this end we need the following simple result.

4.1 LEMMA. *For all $a, b \geq 0$, we have*

$$\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}.$$

PROOF. For all $a, b \geq 0$, we have

$$a + b \leq a + 2\sqrt{a}\sqrt{b} + b = (\sqrt{a} + \sqrt{b})^2,$$

from which the assertion follows by taking the square root of both sides. □

In the following results, $\|w\|_1$ denotes the l_1 norm of a vector w of appropriate dimension. Moreover, we recall that $w^* := (x^*, y^*, z^*) \in \Re^n \times \Re^m \times \Re^m$ is a KKT-point of VIP(X, F) if and only if w^* satisfies the system of equations $\Phi(w; 0, 0) = 0$. We are now in a position to prove the following result.

4.2 LEMMA. *Let $w(\mu,\varepsilon) = (x(\mu,\varepsilon), y(\mu,\varepsilon), z(\mu,\varepsilon))$ be a KKT-point of PVIP(X, F, μ, ε). Then*

$$\|\Phi(w(\mu,\varepsilon); 0, 0)\|_1 \leq 2m\sqrt{\mu} + \varepsilon\|x(\mu,\varepsilon)\|_1.$$

PROOF. Since $w(\mu,\varepsilon)$ is a KKT-point of PVIP(X,F,μ,ε), we have $\Phi(w(\mu,\varepsilon);\mu,\varepsilon)=0$. From the very definition of $\Phi(.;\mu,\varepsilon)$ and $\Phi(.;0,0)$ we therefore obtain

$$\begin{aligned}
\|\Phi(w(\mu,\varepsilon);0,0)\|_1 &= \|\Phi(w(\mu,\varepsilon);0,0)-\Phi(w(\mu,\varepsilon);\mu,\varepsilon)\|_1 \\
&= \sum_{i=1}^m \varepsilon|x_i(\mu,\varepsilon)| + \\
&\quad \sum_{i=1}^m \left|\sqrt{(y_i(\mu,\varepsilon)-z_i(\mu,\varepsilon))^2+4\mu}-\sqrt{(y_i(\mu,\varepsilon)-z_i(\mu,\varepsilon))^2}\right| \\
&\le \varepsilon\sum_{i=1}^m |x_i(\mu,\varepsilon)| + 2\sum_{i=1}^m \sqrt{\mu} \\
&= 2m\sqrt{\mu}+\varepsilon\|x(\mu,\varepsilon)\|_1,
\end{aligned}$$

where the inequality follows from Lemma 4.1. □

4.3 LEMMA. *Let the assumptions of Theorem 3.5 be satisfied, in particular, let w^* denote the unique KKT-point of VIP(X,F). Then, for all $\mu>0$ and $\varepsilon>0$ sufficiently small, there exists a constant $c_1>0$, independent of μ and ε, such that*

$$c_1\|w(\mu,\varepsilon)-w^*\|_1 \le 2m\sqrt{\mu}+\varepsilon\|x(\mu,\varepsilon)\|_1.$$

PROOF. By Lemma 3.4, Theorem 3.5 and Proposition 3 in [24], there exists a constant $c_1>0$ such that

$$c_1\|w-w^*\|_1 \le \|\Phi(w;0,0)\|_1$$

for all w sufficiently close to w^*. Hence we obtain the desired inequality from Lemma 4.2 and the fact that $w(\mu,\varepsilon)$ converges to w^* for $\mu,\varepsilon\to 0$ because of Corollary 3.8. □

4.4 THEOREM. *Let the assumptions of Theorem 3.5 be satisfied. Then there exists a constant $c_2>0$ such that*

$$\|w(\mu,\varepsilon)-w^*\|_1 \le c_2(\sqrt{\mu}+\varepsilon)$$

for all $\mu>0$ and $\varepsilon>0$ sufficiently small.

PROOF. The assertion is an immediate consequence of Lemma 4.3 and the fact that $x(\mu,\varepsilon)$ remains bounded in view of Corollary 3.8. □

Theorem 4.4 states that if we solve the neighbouring problem PVIP(X,F,μ,ε), the solution can be made arbitrarily close to the solution of the original problem VIP(X,F) by taking μ and ε sufficiently small. Note that there are a number of recently proposed "smoothing methods" where a nonsmooth formulation of certain problems (minmax problem, convex optimization problem, complementarity problem) is replaced by a neighbouring smooth problem which is solved instead of the original one. Numerical results presented, e.g., by Chen and Mangasarian [3] and Pinar and Zenios [25] are quite promising. According to Theorem 4.4, we can view our method also as a smoothing method by fixing the perturbation parameters μ and ε and applying, e.g., Newton's method to the system of equations $\Phi(w;\mu,\varepsilon)=0$. In this paper, however, we interprete our method as a continuation method.

5 Numerical Results

5.1 Algorithm

In this section, we present an implementable version of Algorithm 3.3 and its numerical results for problems arising in variational inequalities, nonlinear programming and complementarity problems. In step (S.2) of Algorithm 3.3, a system of nonlinear equations $\Phi(w;\mu,\varepsilon)=0$ has to be solved. Instead of solving this system exactly, we use a damped Newton method in order to find an approximate solution of $\Phi(w;\mu,\varepsilon)=0$. Note that the Jacobian matrices of $\Phi(.;\mu,\varepsilon)$ are nonsingular under the assumptions of Theorem 3.12.

The following algorithm is an implementable version of Algorithm 3.3.

5.1 Algorithm.

(S.0) Choose $w^0 := (x^0, y^0, z^0) \in \Re^n \times \Re^m \times \Re^m$, $\theta \geq 0$, $\mu_0 > 0, \varepsilon_0 > 0$, $\beta, \sigma \in (0,1)$, and set $k := 0$.

(S.1) If $\|\Phi(w^k;0,0)\| \leq \theta$, stop: w^k is an approximate solution of VIP(X,F).

(S.2) Compute $\Delta w^k = (\Delta x^k, \Delta y^k, \Delta z^k) \in \Re^n \times \Re^m \times \Re^m$ as the solution of the linear system

$$\nabla\Phi(w(\mu_k,\varepsilon_k);\mu_k,\varepsilon_k)\Delta w^k = -\Phi(w(\mu_k,\varepsilon_k);\mu_k,\varepsilon_k).$$

(S.3) Let $t^k = \beta^{m_k}$, where m_k is the smallest nonnegative integer m satisfying the following inequality

$$\|\Phi(w^k + \beta^m \Delta w^k;\mu_k,\varepsilon_k)\|^2 \leq (1-\sigma\beta^m)\|\Phi(w(\mu_k,\varepsilon_k);\mu_k,\varepsilon_k)\|^2.$$

Let $w^{k+1} = w^k + t^k \Delta w^k$.

(S.4) Compute μ_{k+1} and ε_{k+1}. Set $k := k+1$, and go to (S.1).

We next describe the computation of the perturbation parameters μ_{k+1} and ε_{k+1} We begin with the updating rule for μ_{k+1} which is very similar to the one proposed by Chen and Harker [1], see also [15].

Updating Rule for μ_k:

(a) Let $u_{k+1} := \||\Phi(w^k; \mu_k, \varepsilon_k)\|/(n+2m)$. If $u_{k+1} \geq 1$, then $\mu_{k+1} := \sqrt{u_{k+1}}$, else $\mu_{k+1} := u_{k+1}$.

(b) If $\mu_{k+1} < 10^{-10}$, then $\mu_{k+1} = 10^{-10}$.

(c) If $\mu_{k+1} > \mu_k$, set $\mu_{k+1} = \mu_k$

(d) If $\||\Phi(w^{k+1}; \mu_k, \varepsilon_k)\| < 10^{-4}$, then $\mu_{k+1} = 10^{-2}\mu_{k+1}$.

For the update of ε_{k+1}, we use the simple rule $\varepsilon_{k+1} = \alpha\mu_{k+1}$, where α is a positive constant (note that this updating rule is motivated by Theorem 3.11).

Algorithm 5.1 has been implemented in Fortran and run on a DEC 5000 workstation. Throughout the computational experiments, the parameters used were set as $\beta = 0.5$, $\sigma = 10^{-4}$, $\theta = 10^{-6}$, $\alpha = 1$, $\varepsilon_0 = 10^{-4}$, and $\mu_0 = \min\{10^{-2}, \|\Phi(w^0; 0, 0)\|\}$. In the numerical results to be reported, we give for each problem the dimension of the system $\Phi(w; \mu, \varepsilon)$ by DIM, the starting point by SP, the number of iterations by IT, the number of evaluations of F by NF (which is equal to the number of evaluations of g, and for the nonlinear programming problems, it is the number of evaluations of the gradient of the objective function), and the final residual $\|\Phi(w; 0, 0)\|$ by ERROR with w representing the final approximate solution of the problem.

5.2 Variational inequality problems

The numerical experiments conducted consist of three parts, namely, variational inequality, nonlinear programming and complementarity problems. The test problems were chosen from the literature without giving full details but some relevant comments. The source for each problem is not necessarily the original one. We begin with variational inequality problems.

VI1 PROBLEM: See [29]. F is a strongly monotone and nonlinear mapping and X is polyhedral. The problem has a parameter ρ which reflects the degree of asymmetry and nonlinearity. The optimal solution is $(2, 2, 2, 2, 2)$.

VI2 PROBLEM: See [29]. This problem is similar to the last problem with the features that F is a strongly monotone and nonlinear mapping and X is polyhedral. We tested one case in which the original dimension of the variational inequality problem is 5. An approximate solution is $(9.08, 4.84, 0, 0, 5)$.

VI3 PROBLEM: See [7]. Again, F is a strongly monotone and nonlinear mapping, but X is not polyhedral. An approximate solution found by our algorithm is $(1, 0, 0)$.

PROBLEM	DIM	SP	IT	NF	ERROR
VI1 ($\rho = 10$)	12	$(1, \cdots, 1)$	5	6	0.59×10^{-6}
VI1 ($\rho = 10^3$)	12	$(1, \cdots, 1)$	8	9	0.22×10^{-6}
VI1 ($\rho = 10^5$)	12	$(1, \cdots, 1)$	8	9	0.29×10^{-6}
VI2	18	$(1, \cdots, 1)$	10	13	0.45×10^{-6}
VI3	5	$(1, \cdots, 1)$	7	8	0.42×10^{-6}
VI4	12	$(1, \cdots, 1)$	6	7	0.18×10^{-6}

Table 1: Results for varitional inequality problems using Algorithm 5.1

VI4 PROBLEM: See [7]. F is an affine and strongly monotone mapping, and X is polyhedral. The solution of this problem is known to be $x^* = (120, 90, 0, 70, 50)$.

The numerical results for the above four variational inequality problems are given in Table 1.

5.3 Nonlinear programming problems

In this subsection, we present the numerical experiments of Algorithm 5.1 being applied to the convex constrained programming problems 3, 10, 11, 12, 14, 21, 22, 28, 34, 35, 43, 48, 49, 50, 51, 52, 53, 55, 65, 66, 73, 76 and 100 from the book of Hock and Schittkowski [11]. The KKT-conditions for each of these problems is a mixed complementarity problem which is a special case of a variational inequality problem. Note that, since we only consider convex optimization problems, the corresponding variational inequality problem VIP$(X, \nabla f)$ (f being the objective function of the optimization problem) is monotone.

Table 2 contains the numerical results for the above nonlinear programming problems. A failure of the algorithm is indicated by the letter “F”.

5.4 Complementarity problems

Finally, we tested some complementarity problems.

KOJIMA-JOSEPHY PROBLEM: See [5]. This is a nonmonotone nonlinear complementarity problem, which has a unique solution.

WATSON SECOND PROBLEM: See [32]. This is a nonmonotone linear complementarity problem.

WATSON FOURTH PROBLEM: See [32]. This is a nonlinear complementarity problem, which represents the KKT conditions for a convex programming problem.

PROBLEM	DIM	SP	IT	NF	ERROR
HS3	3	$(1, \cdots, 1)$	5	6	0.26×10^{-6}
HS10	4	$(1, \cdots, 1)$	10	12	0.18×10^{-7}
HS11	4	$(1, \cdots, 1)$	7	8	0.21×10^{-7}
HS12	4	$(1, \cdots, 1)$	8	14	0.54×10^{-7}
HS14	5	$(1, \cdots, 1)$	6	7	0.57×10^{-6}
HS21	12	$(1, \cdots, 1)$	9	16	0.17×10^{-7}
HS22	6	$(1, \cdots, 1)$	7	8	0.10×10^{-6}
HS28	4	$(1, \cdots, 1)$	3	4	0.87×10^{-12}
HS34	16	$(1, \cdots, 1)$	19	34	0.35×10^{-6}
HS35	8	$(1, \cdots, 1)$	6	7	0.19×10^{-7}
HS43	10	$(1, \cdots, 1)$	F	F	F
HS48	7	$(1, \cdots, 1)$	3	4	0.22×10^{-11}
HS49	7	$(1, \cdots, 1)$	52	136	0.95×10^{-6}
HS50	8	$(1, \cdots, 1)$	23	24	0.96×10^{-6}
HS51	8	$(1, \cdots, 1)$	3	5	0.22×10^{-11}
HS52	8	$(1, \cdots, 1)$	3	4	0.69×10^{-12}
HS53	28	$(1, \cdots, 1)$	5	8	0.29×10^{-7}
HS55	22	$(1, \cdots, 1)$	F	F	F
HS65	17	$(1, \cdots, 1)$	13	36	0.32×10^{-6}
HS66	16	$(1, \cdots, 1)$	15	24	0.30×10^{-6}
HS73	13	$(1, \cdots, 1)$	13	15	0.31×10^{-7}
HS76	14	$(1, \cdots, 1)$	7	8	0.16×10^{-7}
HS100	15	$(1, \cdots, 1)$	14	22	0.10×10^{-7}

Table 2: Results for convex programming problems using Algorithm 5.1

Mathiesen problem: See [19]. This is a Walrasian equilibrium problem. It depends on three parameters α, b_2 and b_3. We report numerical results for the following two choices of these parameters: (a) $\alpha = 0.75, b_2 = 1, b_3 = 0.5$ and (b) $\alpha = 0.75, b_2 = 1, b_3 = 2$. In both cases, the problem has infinitely many solutions.

Modified Mathiesen problem: See [14]. This is a nonmonotone nonlinear complementarity problem reformulated from Mathiesen's Walrasian equilibrium problem. It has the infinitely many solutions $(\lambda, 0, 0, 0)$, where $\lambda \in [0, 3]$.

Nash equilibrium problem: See [9]. This is a nonlinear complementarity problem, in which the associated function is not twice differentiable, but it is a P-function. We tested two cases with the dimensions 5 and 10 of the original problem.

PROBLEM	DIM	SP	IT	NF	ERROR
Kojima-Josephy	8	$(1,\cdots,1)$	8	16	0.73×10^{-6}
Watson 2	10	$(1,\cdots,1)$	6	7	0.24×10^{-6}
Watson 4	10	$(1,\cdots,1)$	23	24	0.93×10^{-6}
Mathiesen (a)	8	$(1,\cdots,1)$	10	11	0.28×10^{-7}
Mathiesen (b)	8	$(1,\cdots,1)$	8	10	0.30×10^{-7}
mod. Mathiesen	8	$(1,\cdots,1)$	6	9	0.47×10^{-6}
Nash equilibrium	10	$(1,\cdots,1)$	10	11	0.26×10^{-6}
	20	$(1,\cdots,1)$	9	10	0.22×10^{-6}
Spatial equilibrium	84	$(1,\cdots,1)$	27	39	0.43×10^{-6}

Table 3: Results for complementarity problems using Algorithm 5.1

SPATIAL PRICE EQUILIBRIUM PROBLEM: See [30]. This is a problem arising in a spatial equilibrium model. The dimension of the original problem is 42.

The numerical results for the above complementarity problems are listed in Table 3. The numerical results given in Tables 1 – 3 are quite promising, and most problems were solved using just a small number of iterations although the test problems came up from different mathematical areas. We note that we have two failures, namely for the problems HS 43 and HS 55 in Table 2, but that we used the same parameter settings for all test runs without optimizing them for specific problems.

6 Final Remarks

In this paper, we presented a new continuation method for the solution of monotone variational inequality problems $\mathrm{VIP}(X,F)$. The central idea was to reformulate the optimality conditions of $\mathrm{VIP}(X,F)$ into an equivalent system of nonlinear equations. This reformulation is based on the function φ_μ defined in (1) and introduced by one of the authors in [15]. Actually, in [15] some other but related functions were introduced, e.g.

$$\psi_\mu(a,b) := a + b - \sqrt{a^2 + b^2 + 2\mu}.$$

For $\mu = 0$, this function reduces to a function introduced by Fischer [6]. It is not difficult to see that all results of this paper remain true if the function φ_μ is replaced by the function ψ_μ everywhere. The only nontrivial part is the result which corresponds to Theorem 3.5, see, however, Jiang [13].

We also made some numerical experiments using the function ψ_μ. But, although the results were still good, the results were slightly inferior to those obtained with the function φ_μ. For this reason, we decided to consider only the function φ_μ in this paper.

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