A New Approach to Continuation Methods for Complementarity Problems with Uniform $P-$Functions

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Abstract. We design a new continuation method for the solution of nonlinear complementarity problems with uniform $P-$functions. Similar to interior-point methods, we try to follow the central path inexactly. In contrast to interior-point methods, however, our iterates are allowed to stay outside of the positive orthant. The method is shown to be globally and superlinearly (quadratically) convergent.

Key words. nonlinear complementarity problems, nonlinear equations, continuation methods, interior-point methods, uniform $P-$functions, P_0-functions.

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Abbreviated title. Continuation Methods for Complementarity Problems.

1 Introduction

In this paper, we consider the nonlinear complementarity problem. This problem arises in many applications, e.g., in operations research, economic equilibrium models and in the engineering sciences (contact problems, obstacle problems, ...), see [11, 5] for a more detailed description. Several methods for the solution of complementarity problems are known. Among them are reformulations of the complementarity problem as a fixed-point problem, as an optimization problem and as a (smooth or nonsmooth) system of nonlinear equations. We refer the reader to the papers [11, 9] for a survey of several of these methods.

In recent years, the interior-point approach has also been generalized from the linear and quadratic programming problem to the (linear and) nonlinear complementarity problem. For example, Kojima et al. [16, 17, 14] prove some results which can be viewed as a theoretical foundation of continuation methods for nonlinear complementarity problems. In particular, they consider complementarity problems with a uniform $P-$function, see [16]. This is exactly the problem also considered in this paper. The approach given here, however, is completely different from the one in [16]. Based on a tool recently introduced by the author in [13], a certain perturbed complementarity problem is reformulated as a nonlinear system of equations. We first use this tool in order to obtain a convergence result, and then show how this tool can be used numerically, leading to a noninterior continuation method where the iterates do not necessarily have to stay in the interior of the feasible region during an entire iteration. This is an interesting difference to commonly used interior-point methods, and some promising numerical results are reported in [13] for the linear complementarity problem.

A similar algorithm has recently been proposed by Chen and Harker [1], but only for the linear complementarity problem and under some different assumptions, see also Chen and Harker [2, 3] for related methods in another context. As noted in [13], the method by Chen and Harker [1] can be shown to be a special case of our approach if the assumptions used in [1] are satisfied.

The paper is organized as follows: In Section 2, we present some background material. The continuation method itself is introduced in Section 3. We note that it is well–defined and prove that any sequence generated by this method converges to the unique solution of the underlying complementarity problem. We also present an inexact version of the algorithm and give a global and local convergence result for it. We conclude this paper with some final remarks in Section 4.

Throughout this paper, the index set $\{1, \ldots, n\}$ is abbreviated by I. The n-dimensional real space is denoted by $\Re^n$. If $x, y \in \Re^n$ are two vectors, then the vector $(x^T, y^T)^T \in \Re^{2n}$ is simply written as (x, y). Inequalities such as $x \geq 0$ and $x > 0$ are defined componentwise. The nonnegative orthant of $\Re^n$ is denoted by $\Re^n_+$. $\|z\|$ always denotes the Euclidean norm of a given vector z of appropriate dimension. For a matrix $M \in \Re^{n \times n}, M = (m_{ij})$, and an index set $J \subseteq I$, the submatrix M_{JJ} consists of the elements $m_{ij}, i, j \in J$.

2 Background Material

The problem under investigation is the following.

Definition 2.1 *The nonlinear complementarity problem, denoted by NCP(F), is to find a vector pair* $(x^*, y^*) \in \Re^{2n}$ *satisfying the conditions*

$$x \geq 0, y \geq 0, x^T y = 0, y = F(x). \tag{1}$$

If $F(x) = Mx + q$ *is an affine function, where* $M \in \Re^{n \times n}$ *and* $q \in \Re^n$*, then NCP(F) is called a linear complementarity problem and is denoted by LCP*(q, M)*.*

We would like to solve the nonlinear complementarity problem by successive solution of certain perturbed complementarity problems which were introduced by McLinden [18] and which are defined in

Definition 2.2 *Let* $\mu \geq 0$ *be any given parameter. The perturbed nonlinear complementarity problem, PNCP*(F, μ) *for short, is to find a solution* $(x(\mu), y(\mu)) \in \Re^{2n}$ *of the following system:*

$$x \geq 0, y \geq 0, x_i y_i = \mu \ \ (i \in I), y = F(x).$$

If $\mu = 0$ then problem PNCP(F, μ) reduces to the nonlinear complementarity problem (1). – Some classes of functions F which are important in the investigation of NCP(F) are introduced in the next definition.

Definition 2.3 *Let* $F : \Re^n \to \Re^n$*. The mapping* F *is said to be a*

(a) P_0-*function if for all* $x, y \in \Re^n, x \neq y$*, there exists an index* $i \in I$ *such that* $x_i \neq y_i$ *and*

$$(x_i - y_i)(F_i(x) - F_i(y)) \geq 0.$$

(b) uniform $P-$*function (with modulus* $\gamma > 0$*) if*

$$\max_{i \in I}(x_i - y_i)(F_i(x) - F_i(y)) \geq \gamma \|x - y\|^2 \quad \forall x, y \in \Re^n.$$

If F is an affine function, i.e., $F(x) = Mx + q$ for some matrix $M \in \Re^{n \times n}$ and some vector $q \in \Re^n$, then M is called a $P-$*matrix* if F is a uniform $P-$function. Similarly, M is called a P_0-*matrix* if F is a P_0-function.

The following result is due to Moré [19, Theorem 2.3].

Lemma 2.4 *Let* $F : \Re^n \to \Re^n$ *be a continuous and uniform* $P-$*function. Then problem NCP(F) has a unique solution.*

The proof of the next result can be found in Moré and Rheinboldt [20, Corollary 5.3 and Theorem 5.8].

Lemma 2.5 *Let* $F : \Re^n \to \Re^n$ *be continuously differentiable. Then* $F'(x)$ *is a* P_0-*matrix for all* $x \in \Re^n$ *if and only if* F *is a* P_0-*function.*

3 Continuation Method

Let $\mu \geq 0$ be given. The main tool used in this paper is the function $\varphi_\mu : \Re^2 \to \Re$ defined by

$$\varphi_\mu(a,b) := a + b - \sqrt{(a-b)^2 + 4\mu}. \tag{2}$$

This function has recently been introduced by the author in [13], where the following lemma is proved.

Lemma 3.1 *The function φ_μ has the property*

$$\varphi_\mu(a,b) = 0 \Longleftrightarrow a \geq 0, b \geq 0, ab = \mu.$$

In the special case $\mu = 0$, we note that $\varphi_\mu = \varphi_0$ reduces to

$$\varphi_0(a,b) = a + b - \sqrt{(a-b)^2} = a + b - |a-b| = 2\min\{a,b\}. \tag{3}$$

The function $\min\{a,b\}$ has been used, e.g., by Pang [21] in order to characterize problem NCP(F). Here, the function φ_μ is used to characterize problem PNCP(F, μ). To this end, let us define the nonlinear operator $F_{\varphi_\mu} : \Re^{2n} \to \Re^{2n}$ by

$$F_{\varphi_\mu}(z) := F_{\varphi_\mu}(x,y) := \begin{pmatrix} F(x) - y \\ \varphi_\mu(x,y) \end{pmatrix}, \tag{4}$$

where

$$\varphi_\mu(x,y) := (\varphi_\mu(x_1,y_1), \ldots, \varphi_\mu(x_n,y_n))^T \in \Re^n.$$

Using Lemma 3.1, we directly obtain the following

Theorem 3.2 *A vector $z(\mu) := (x(\mu), y(\mu)) \in \Re^{2n}$ solves the perturbed nonlinear complementarity problem PNCP(F, μ) if and only if $z(\mu)$ solves the nonlinear system of equations $F_{\varphi_\mu}(z) = 0$.*

This result motivates the following algorithm.

Algorithm 3.3 *(Continuation method)*

(S.0): Let $\{\mu_k\}$ be any sequence such that $\mu_k > 0$ and $\lim_{k\to\infty} \mu_k = 0$. Set $k = 0$.

(S.1): Find a solution $z^k := z(\mu_k)$ of the nonlinear system of equations

$$F_{\varphi_{\mu_k}}(z) = 0.$$

(S.2): If $\|F_{\varphi_0}(z^k)\| = 0$, stop: z^k solves NCP(F).

(S.3): Set $k := k + 1$ and go to (S.1).

Before giving a convergence result, we first restate a lemma due to Kojima et al. [15, Lemma 4.1].

Lemma 3.4 *The matrix*

$$\begin{pmatrix} M & -I_n \\ D_a & D_b \end{pmatrix}$$

is nonsingular for any positive definite diagonal matrices $D_a, D_b \in \Re^{n\times n}$ *if and only if* $M \in \Re^{n\times n}$ *is a* P_0-*matrix.*

Based on this lemma, we are able to establish the following theorem.

Theorem 3.5 *Let* $\mu > 0, F : \Re^n \to \Re^n$ *be continuously differentiable and define* F_{φ_μ} *as in (4). Then the Jacobian matrix* $F'_{\varphi_\mu}(z)$ *is nonsingular for all* $z := (x, y) \in \Re^{2n}$ *if* F *is a* P_0-*function.*

Proof. We first note that the function φ_μ is continuously differentiable for all $\mu > 0$. Therefore the Jacobian $F'_{\varphi_\mu}(z)$ exists and is given by

$$F'_{\varphi_\mu}(z) = \begin{pmatrix} F'(x) & -I_n \\ D_a & D_b \end{pmatrix},$$

where

$$\begin{aligned} D_a := D_a(z) &:= diag\left(\frac{\partial\varphi_\mu}{\partial a}(x_1, y_1), \ldots, \frac{\partial\varphi_\mu}{\partial a}(x_n, y_n)\right), \\ D_b := D_b(z) &:= diag\left(\frac{\partial\varphi_\mu}{\partial b}(x_1, y_1), \ldots, \frac{\partial\varphi_\mu}{\partial b}(x_n, y_n)\right). \end{aligned}$$

Since $\frac{\partial\varphi_\mu}{\partial a}(a, b) \in (0, 2)$ and $\frac{\partial\varphi_\mu}{\partial b}(a, b) \in (0, 2)$ for all $\mu > 0$ and all $(a, b) \in \Re^2$, the diagonal matrices D_a and D_b are positive definite for all $z \in \Re^{2n}$. We therefore obtain from Lemma 3.4 that $F'_{\varphi_\mu}(z)$ is nonsingular for all $z = (x, y) \in \Re^{2n}$ if $F'(x)$ is a P_0-matrix for all $x \in \Re^n$. Because of Lemma 2.5 the latter condition is equivalent to F being a P_0-function. This proves the desired result. □

Lemma 3.4 raises the interesting question whether or not the P_0-function property in Theorem 3.5 is also necessary for the Jacobian matrices $F'_{\varphi_\mu}(z)$ to be nonsingular. The author is currently not certain whether it is always possible to find a vector $z = (x, y) \in \Re^{2n}$ in such a way that $F'_{\varphi_\mu}(z)$ becomes singular for functions F which are not P_0-functions. (Note that the diagonal matrices D_a and D_b as defined in the proof of Theorem 3.5 cannot be chosen independently.) In the following analysis, however, only the sufficiency part given in Theorem 3.5 is needed.

We next restate a result which follows from the results by Kojima, Mizuno and Noma [16].

Theorem 3.6 *Let $F : \Re^n \to \Re^n$ be a continuous and uniform $P-$function. Then the perturbed problem PNCP(F, μ) has a unique solution $z(\mu)$ for each $\mu > 0$.*

In view of Theorem 3.6, the sequence $\{z^k\}$ as generated by Algorithm 3.3 is well-defined. We next want to show that this sequence converges to the unique solution z^* of the original problem NCP(F). To this end, we first prove some simple properties of the function φ_μ.

Lemma 3.7 *Let $\mu, \mu_1, \mu_2 \geq 0$ be arbitrarily given. Then the following hold:*

(a) $|\varphi_{\mu_1}(a,b) - \varphi_{\mu_2}(a,b)| \leq 2|\sqrt{\mu_1} - \sqrt{\mu_2}|$ *for all* $(a,b) \in \Re^2$.

(b) If $\{a_k\}, \{b_k\} \subseteq \Re$ *are two sequences with* $|a_k| \to \infty$ *and* $|b_k| \to \infty$*, then* $|\varphi_\mu(a_k, b_k)| \to \infty$.

Proof. Consider part (a). If $\mu_1 = \mu_2 = 0$, there is nothing to prove. Hence assume that at least one of the perturbation parameters is positive. Then we get, for any $(a,b) \in \Re^2$:

$$\begin{aligned} |\varphi_{\mu_1}(a,b) - \varphi_{\mu_2}(a,b)| &= \left|\sqrt{(a-b)^2 + 4\mu_1} - \sqrt{(a-b)^2 + 4\mu_2}\right| \\ &= \frac{4|\mu_1 - \mu_2|}{\left|\sqrt{(a-b)^2 + 4\mu_1} + \sqrt{(a-b)^2 + 4\mu_2}\right|} \\ &\leq \frac{2|\mu_1 - \mu_2|}{|\sqrt{\mu_1} + \sqrt{\mu_2}|} \\ &= 2|\sqrt{\mu_1} - \sqrt{\mu_2}|. \end{aligned}$$

Assertion (b) can easily be verified, see also [12]. □

The following important boundedness result is the main step in the proof of our convergence result (see Theorem 3.9 below).

Theorem 3.8 *Let $F : \Re^n \to \Re^n$ be a continuous and uniform $P-$function. Then the level sets*

$$\mathcal{L}(\mu, \alpha) := \{(x,y) \in \Re^{2n} |\ \|F_{\varphi_\mu}(x,y)\| \leq \alpha\} \tag{5}$$

are uniformly bounded for all $0 \leq \alpha \leq \bar{\alpha} < \infty$ *and* $0 \leq \mu \leq \bar{\mu} < \infty$.

Proof. The proof is an extension of the one given in [10, Theorem 3.2]. Suppose there exists an unbounded sequence $\{z^k\} := \{(x^k, y^k)\}$. Let $z^k \in \mathcal{L}(\mu_k, \alpha_k)$ for some $0 \leq \mu_k \leq \bar{\mu}, 0 \leq \alpha_k \leq \bar{\alpha}$. Without loss of generality we can assume that $\mu_k \to \mu_*$ and $\alpha_k \to \alpha_*$ for certain $0 \leq \mu_* \leq \bar{\mu}$ and $0 \leq \alpha_* \leq \bar{\alpha}$. Since the sequence $\{z^k\} = \{(x^k, y^k)\}$ is unbounded and

$$|F_i(x^k) - y_i^k| \leq \|F(x^k) - y^k\| \leq \|F_{\varphi_{\mu_k}}(z^k)\| \leq \alpha_k \leq \bar{\alpha}, \tag{6}$$

we obtain that the sequence $\{x^k\}$ must be unbounded. Consequently, the index set

$$J := \{i \in I |\, \{x_i^k\} \text{ is unbounded}\}$$

is nonempty. Let us define a second sequence $\{\tilde{x}^k\} \subseteq \Re^n$ by

$$\tilde{x}_i^k := \begin{cases} 0 & \text{if } i \in J, \\ x_i^k & \text{if } i \notin J. \end{cases}$$

By construction, the sequence $\{\tilde{x}^k\}$ is bounded. Furthermore, using the definition of $\tilde{x}^k$ and the uniform $P-$function property of F, we obtain with some constant $\gamma > 0$:

$$\begin{aligned} \gamma \sum_{i\in J} (x_i^k)^2 &= \gamma \|x^k - \tilde{x}^k\|^2 \\ &\leq \max_{i\in I} (x_i^k - \tilde{x}_i^k)(F_i(x^k) - F_i(\tilde{x}^k)) \\ &= \max_{i\in J} x_i^k (F_i(x^k) - F_i(\tilde{x}^k)) \\ &\leq \sqrt{\sum_{i\in J} (x_i^k)^2} \sum_{i\in J} |F_i(x^k) - F_i(\tilde{x}^k)|. \end{aligned}$$

Since $\sum_{i\in J}(x_i^k)^2 \neq 0$ at least on a subsequence $\{x^k\}_{k\in K_1}$, we therefore have

$$\gamma \sqrt{\sum_{i\in J} (x_i^k)^2} \leq \sum_{i\in J} |F_i(x^k) - F_i(\tilde{x}^k)|.$$

Due to the boundedness of $\{\tilde{x}^k\}$ and the continuity of $F_i \ (i \in J)$, we thus obtain $|F_{i_0}(x^k)| \to \infty \ (k \in K_1)$ for at least one index $i_0 \in J$. Because of (6), this implies $|y_{i_0}^k| \to \infty \ (k \in K_1)$. However, since $i_0 \in J$, we also have $|x_{i_0}^k| \to \infty$ on a subsequence $\{x_{i_0}^k\}_{K_2}, K_2 \subseteq K_1$. From Lemma 3.7 (b) we therefore get

$$|\varphi_{\mu_*}(x_{i_0}^k, y_{i_0}^k)| \to \infty \quad (k \in K_2). \tag{7}$$

We now consider the behaviour of the sequence $\{\varphi_{\mu_k}(x_{i_0}^k, y_{i_0}^k)\}_{K_2}$. First note that this sequence is bounded since

$$|\varphi_{\mu_k}(x_{i_0}^k, y_{i_0}^k)| \leq \|F_{\varphi_{\mu_k}}(x^k, y^k)\| \leq \alpha_k \leq \bar{\alpha}. \tag{8}$$

On the other hand, we have

$$\begin{aligned} |\varphi_{\mu_k}(x_{i_0}^k, y_{i_0}^k)| &= |\varphi_{\mu_k}(x_{i_0}^k, y_{i_0}^k) - \varphi_{\mu_*}(x_{i_0}^k, y_{i_0}^k) + \varphi_{\mu_*}(x_{i_0}^k, y_{i_0}^k)| \\ &\geq \Big| |\varphi_{\mu_k}(x_{i_0}^k, y_{i_0}^k) - \varphi_{\mu_*}(x_{i_0}^k, y_{i_0}^k)| - |\varphi_{\mu_*}(x_{i_0}^k, y_{i_0}^k)| \Big| \\ &\geq |\varphi_{\mu_*}(x_{i_0}^k, y_{i_0}^k)| - |\varphi_{\mu_k}(x_{i_0}^k, y_{i_0}^k) - \varphi_{\mu_*}(x_{i_0}^k, y_{i_0}^k)|. \end{aligned} \tag{9}$$

Since $\mu_k \to \mu_*$, we obtain from Lemma 3.7 (a) that $|\varphi_{\mu_k}(x_{i_0}^k, y_{i_0}^k) - \varphi_{\mu_*}(x_{i_0}^k, y_{i_0}^k)| \to 0$. Using (7), this yields $|\varphi_{\mu_k}(x_{i_0}^k, y_{i_0}^k)| \to \infty$, a contradiction to (8). Therefore the sequence $\{z^k\}$ is bounded. □

We now state our main convergence result for Algorithm 3.3.

Theorem 3.9 *Let $F : \Re^n \to \Re^n$ be a continuously differentiable and uniform $P-$function. Then the sequence $\{z(\mu_k)\}$ generated by Algorithm 3.3 converges to the unique solution of NCP(F).*

Proof. Because of Theorem 3.6, the sequence $\{z(\mu_k)\}$ is well–defined. Since $z^k = z(\mu_k)$ is a solution of $F_{\varphi_{\mu_k}}(z) = 0$, we have $\|F_{\varphi_{\mu_k}}(z^k)\| = 0$ for all k. Therefore and because the sequence $\{\mu_k\}$ remains bounded in view of $\mu_k \in [0, \mu_0]$ for all k, it follows immediately from Theorem 3.8 that the sequence $\{z^k\}$ is also bounded. Consequently there exists at least one accumulation point z^*. From the continuity of F we directly obtain that z^* is a solution of NCP(F). This shows that every accumulation point of $\{z(\mu_k)\}$ is a solution of NCP(F). However, as noted in Lemma 2.4, problem NCP(F) has a unique solution under the stated assumptions. Therefore the entire sequence $\{z(\mu_k)\}$ converges to this solution. □

In our next theorem we show that z^{k-1} is a good starting vector for, e.g., Newton's method when solving the nonlinear system of equations $F_{\varphi_{\mu_k}}(z) = 0$ in step (S.1) of Algorithm 3.3. We first restate a technical lemma which is due to Kojima, Mizuno and Noma [16, Lemma 1].

Lemma 3.10 *For any nonnegative numbers $\alpha^1, \alpha^2, \beta^1$ and β^2, the following inequality holds:*

$$(\alpha^1 - \alpha^2)(\beta^1 - \beta^2) \leq |\alpha^1\beta^1 - \alpha^2\beta^2|.$$

Theorem 3.11 *If $F : \Re^n \to \Re^n$ is Lipschitz–continuous and a uniform $P-$function and if $\{z^k\}$ denotes the sequence generated by Algorithm 3.3, then there exists a constant $c > 0$ (independent of k) such that*

$$\|z^{k+1} - z^k\|^2 \leq c|\mu_{k+1} - \mu_k|$$

for all k.

Proof. Let $L > 0$ denote the Lipschitz–constant of F. Then, we have

$$\begin{aligned} \|z^{k+1} - z^k\|^2 &= \|x^{k+1} - x^k\|^2 + \|y^{k+1} - y^k\|^2 \\ &= \|x^{k+1} - x^k\|^2 + \|F(x^{k+1}) - F(x^k)\|^2 \\ &\leq (1 + L^2)\|x^{k+1} - x^k\|^2. \end{aligned} \tag{10}$$

Since F is a uniform $P-$function and z^{k+1} and z^k are solutions of PNCP(F, μ_{k+1}) and PNCP(F, μ_k), respectively, we obtain with some $\gamma > 0$ from Lemma 3.10:

$$\begin{aligned} \gamma\|x^{k+1} - x^k\|^2 &\leq \max_{i \in I}(x_i^{k+1} - x_i^k)(F_i(x^{k+1}) - F_i(x^k)) \\ &= \max_{i \in I}(x_i^{k+1} - x_i^k)(y_i^{k+1} - y_i^k) \\ &\leq \max_{i \in I} |x_i^{k+1}y_i^{k+1} - x_i^k y_i^k| \\ &= |\mu_{k+1} - \mu_k|. \end{aligned} \tag{11}$$

Thus, the assertion follows from (10) and (11) with $c := (1 + L^2)/\gamma$. □

The main disadvantage of Algorithm 3.3 is the fact that we have to solve the subproblems PNCP(F, μ) exactly at each iteration. The following is an inexact version of this algorithm, where we call a vector $z(\varepsilon)$ an ε-*approximate solution* of the system $F_{\varphi_\mu}(z) = 0$ if $\|F_{\varphi_\mu}(z(\varepsilon))\| \leq \varepsilon$.

Algorithm 3.12 *(Inexact continuation method)*

(S.0): Choose $\mu_0 > 0, \varepsilon_0 > 0$ and set $k = 0$.

(S.1): Find an ε_k-approximate solution $z(\varepsilon_k)$ of the system $F_{\varphi_{\mu_k}}(z) = 0$.

(S.2): Terminate the iteration if a suitable stopping criterion is satisfied.

(S.3): Choose $\mu_{k+1} < \mu_k, \varepsilon_{k+1} < \varepsilon_k$, set $k := k + 1$ and go to (S.1).

In the following convergence analysis of Algorithm 3.12, we assume that an infinite sequence $\{z(\varepsilon_k)\}$ is generated. The first result is a global convergence theorem for Algorithm 3.12.

Theorem 3.13 *Let $F : \Re^n \to \Re^n$ be a continuously differentiable and uniform P-function. Assume that the sequences $\{\varepsilon_k\}$ and $\{\mu_k\}$ converge to 0. Then the sequence $\{z(\varepsilon_k)\}$ generated by Algorithm 3.12 converges to the unique solution of NCP(F).*

Proof. The proof is similar to the one of Theorem 3.9: Since both sequences $\{\mu_k\}$ and $\{\varepsilon_k\}$ remain bounded, the sequence $\{z(\varepsilon_k)\}$ is also bounded by Theorem 3.8. Hence there is at least one accumulation point, say z^*. Let $\{z(\varepsilon_k)\}_K$ denote a subsequence converging to z^*. Since $z(\varepsilon_k)$ is an ε_k-approximate solution of $F_{\varphi_{\mu_k}}(z) = 0$, we have

$$\|F_{\varphi_{\mu_k}}(z(\varepsilon_k))\| \leq \varepsilon_k. \tag{12}$$

Since $\mu_k \to 0$ and $\varepsilon_k \to 0$, we obtain from (12) for $k \to \infty, k \in K$:

$$\|F_{\varphi_0}(z^*)\| = 0,$$

i.e., z^* solves NCP(F). Since NCP(F) has a unique solution, the bounded sequence $\{z(\varepsilon_k)\}$ cannot have more than one accumulation point, therefore the whole sequence $\{z(\varepsilon_k)\}$ converges to z^*. □

In order to establish a local rate of convergence result for Algorithm 3.12, we first note that the operator F_{φ_μ} is nonsmooth for $\mu = 0$ but nevertheless locally Lipschitz-continuous. Hence its *generalized Jacobian* $\partial F_{\varphi_0}(z)$ exists at any point $z = (x, y) \in \Re^{2n}$. The interested reader is referred to Clarke [4] for the definition and some basic properties of the generalized Jacobian. Here we just note that $\partial F_{\varphi_0}(z)$ is a set of matrices of dimension $2n$. We can prove the following result which is crucial in order to prove a fast local rate of convergence.

Lemma 3.14 *Assume that $F : \Re^n \to \Re^n$ is continuously differentiable and a uniform $P-$function. Let $z = (x, y) \in \Re^{2n}$ be an arbitrary vector. Then all matrices in the generalized Jacobian $\partial F_{\varphi_0}(z)$ are nonsingular.*

Proof. Since F is a uniform $P-$function, its Jacobian matrices $F'(x)$ are $P-$matrices for all $x \in \Re^n$. Hence the desired result follows from Theorem 3.3 in [8] by noting that φ_μ reduces to a multiple of the minimum-function for $\mu = 0$, cf. (3). □

In the following result, we deal with the local rate of convergence of Algorithm 3.12. The proof is based on a result for so-called *semismooth functions*, see [23, 22] for the definition and some elementary properties of semismooth functions. Here we only note that a piecewise smooth function is semismooth. Hence, since the operator F_{φ_0} is piecewise smooth in view of (3), it is also semismooth.

Theorem 3.15 *Let $F : \Re^n \to \Re^n$ be a continuously differentiable and uniform $P-$function. Let $\{z(\varepsilon_k)\}$ be a sequence generated by Algorithm 3.12. Assume that $\{z(\varepsilon_k)\}$ converges to the unique solution z^* of NCP(F). Suppose further that $\mu_k = O(\varepsilon_k^2)$. Then the following statements hold:*

(a) If $\varepsilon_{k+1} = o(\|F_{\varphi_{\mu_0}}(z(\varepsilon_k))\|)$, then $z(\varepsilon_k) \to z^*$ Q-superlinearly.

(b) If $\varepsilon_{k+1} = O(\|F_{\varphi_{\mu_0}}(z(\varepsilon_k))\|^2)$, then $z(\varepsilon_k) \to z^*$ Q-quadratically.

Proof. Recall that $z(\varepsilon_k)$ is an ε_k-approximate solution of PNCP(F, μ_k), so that (12) holds. From the definition of F_{φ_μ} and Lemma 3.7 (a), we have

$$\begin{array}{rcl} \|F_{\varphi_0}(z) - F_{\varphi_\mu}(z)\| & \leq & c_1 \sum_{i \in I} |\varphi_0(x_i, y_i) - \varphi_\mu(x_i, y_i)| \\ & \leq & 2nc_1\sqrt{\mu} \end{array} \tag{13}$$

for a suitable constant $c_1 > 0$ and all $z = (x, y) \in \Re^{2n}$. Since F_{φ_0} is locally Lipschitz-continuous, there is a Lipschitz-constant $L > 0$ such that

$$\|F_{\varphi_0}(z(\varepsilon_k))\| = \|F_{\varphi_0}(z(\varepsilon_k)) - F_{\varphi_0}(z^*)\| \leq L\|z(\varepsilon_k) - z^*\| \tag{14}$$

for all k sufficiently large since $z(\varepsilon_k) \to z^*$. By Lemma 3.14, all elements in Clarke's generalized Jacobian $\partial F_{\varphi_0}(z^*)$ are nonsingular. Moreover, as noted above, F_{φ_0} is semismooth, so we obtain from Proposition 3 in [22] that there exists a constant $c_2 > 0$ such that

$$c_2\|z(\varepsilon_{k+1}) - z^*\| \leq \|F_{\varphi_0}(z(\varepsilon_{k+1}))\| \tag{15}$$

for all k large enough. Now assume that the assumptions of statement (a) are satisfied. Then we obtain from (12), (13), (14) and (15):

$$\begin{array}{rcl} c_2\|z(\varepsilon_{k+1}) - z^*\| & \overset{(15)}{\leq} & \|F_{\varphi_0}(z(\varepsilon_{k+1}))\| \\ & \leq & \|F_{\varphi_0}(z(\varepsilon_{k+1})) - F_{\varphi_{\mu_{k+1}}}(z(\varepsilon_{k+1}))\| + \|F_{\varphi_{\mu_{k+1}}}(z(\varepsilon_{k+1}))\| \end{array}$$

$$
\begin{aligned}
&\overset{(12),(13)}{\leq} && 2nc_1\sqrt{\mu_{k+1}} + \varepsilon_{k+1} \\
&\leq && o(\|F_{\varphi_0}(z(\varepsilon_k))\|) \\
&\overset{(14)}{\leq} && o(\|z(\varepsilon_k) - z^*\|),
\end{aligned}
$$

i.e., $z(\varepsilon_k) \to z^*$ Q-superlinearly. Statement (b) can be shown in a similar way. □

Note that Algorithm 3.12 is an implementable algorithm and that it is always possible to choose the parameters ε_k and μ_k in a way specified in Theorem 3.15. Actually, Theorem 3.15 can be viewed as a theoretical justification of the heuristic updating rules used in [1, 13]. We close this section by noting that some very promising numerical results for a similar algorithm applied to linear complementarity problems are reported in [13].

4 Final Remarks

The function defined in (2) is not the only one having the property mentioned in Lemma 3.1. In fact, the author introduced in [13] three other functions which share the same property, namely:

$$\varphi_\mu(a,b) := a + b - \sqrt{a^2 + b^2 + 2\mu}, \tag{16}$$

$$\varphi_\mu(a,b) := \frac{1}{2}\min{}^2\{0, a+b\} - ab + \mu, \tag{17}$$

$$\varphi_\mu(a,b) := (a-b)^2 - a|a| - b|b| + 2\mu. \tag{18}$$

It is not difficult to see that all these functions also have the properties stated in Lemma 3.7 (b). Even Lemma 3.7 (a) remains true in a slightly modified version. In particular, if φ_μ denotes the function defined in (16) and $\mu_1, \mu_2 \geq 0$ are arbitrarily given, the following inequality can be verified using similar techniques as used in the proof of Lemma 3.7:

$$|\varphi_{\mu_1}(a,b) - \varphi_{\mu_2}(a,b)| \leq \sqrt{2}|\sqrt{\mu_1} - \sqrt{\mu_2}| \quad \forall (a,b) \in \Re^2.$$

Unfortunately, Theorem 3.5 becomes false for the functions defined in (17) and (18) since the diagonal matrices D_a and D_b introduced in the proof of Theorem 3.5 are in general not positive definite. On the other hand these diagonal matrices are positive definite for all $\mu > 0$ and all $(a,b) \in \Re^2$ for the function defined in (16). So all results in Section 3 remain true if the function (2) is replaced by the function (16) everywhere. Numerically, however, these two functions have a similar behaviour, see the results reported in [13].

While the function (2) reduces to the well–known min–function for $\mu = 0$ (cf. (3)), the function φ_μ defined in (16) coincides for $\mu = 0$ with a function introduced by Fischer and used in some recent papers [6, 7]. Fischer uses his function in order to characterize the linear complementarity problem as well as the Karush–Kuhn–Tucker optimality conditions of a nonlinear program, whereas here (for arbitrary $\mu \geq 0$) it can be used to characterize problem PNCP(F, μ). For $\mu = 0$, the other two functions defined in (17) and (18) also reduce to some known functions, see, e.g, [12].

References

[1] B. Chen and P. T. Harker: *A non–interior–continuation method for linear complementarity problems.* SIAM Journal on Matrix Analysis and Applications 14, 1993, pp. 1168–1190.

[2] B. Chen and P. T. Harker: *A noninterior continuation method for quadratic and linear programming.* SIAM Journal on Optimization 3, 1993, pp. 503–515.

[3] B. Chen and P. T. Harker: *A continuation method for monotone variational inequalities.* Mathematical Programming 69, 1995, pp. 237–253.

[4] F. H. Clarke: *Optimization and Nonsmooth Analysis.* John Wiley & Sons, New York, 1983.

[5] M. C. Ferris and J.–S. Pang: *Engineering and economic applications of complementarity problems.* Technical Report 95-07, Computer Sciences Department, University of Wisconsin, Madison, WI, May 1995.

[6] A. Fischer: *A special Newton–type optimization method.* Optimization 24, 1992, pp. 269–284.

[7] A. Fischer: *A Newton–type method for positive semidefinite linear complementarity problems.* Journal of Optimization Theory and Applications 86, 1995, pp. 585–608.

[8] A. Fischer and C. Kanzow: *On finite termination of an iterative method for linear complementarity problems.* Mathematical Programming, to appear.

[9] M. Fukushima: *Merit functions for variational inequality and complementarity problems.* In G. Di Pillo and F. Giannessi (eds.): *Nonlinear Optimization and Applications*, to appear.

[10] C. Geiger and C. Kanzow: *On the resolution of monotone complementarity problems.* Computational Optimization and Applications 5, 1996, pp. 155–173.

[11] P. T. Harker and J.–S. Pang: *Finite–dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications.* Mathematical Programming 48, 1990, pp. 161–220.

[12] C. Kanzow: *Global convergence properties of some iterative methods for linear complementarity problems.* SIAM Journal on Optimization 6, 1996, pp. 326–341.

[13] C. Kanzow: *Some noninterior continuation methods for linear complementarity problems.* SIAM Journal on Matrix Analysis and Applications, to appear.

[14] M. Kojima, N. Meggido and T. Noma: *Homotopy continuation methods for nonlinear complementarity problems.* Mathematics of Operations Research 16, 1991, pp. 754–774.

[15] M. Kojima, N. Meggido, T. Noma and A. Yoshise: *A Unified Approach to Interior Point Algorithms for Linear Complementarity Problems.* Lecture Notes in Computer Science 538, Springer–Verlag, Berlin, 1991.

[16] M. Kojima, S. Mizuno and T. Noma: *A new continuation method for complementarity problems with uniform $P-$functions.* Mathematical Programming 43, 1989, pp. 107–113.

[17] M. Kojima, S. Mizuno and T. Noma: *Limiting behavior of trajectories generated by a continuation method for monotone complementarity problems.* Mathematics of Operations Research 15, 1990, pp. 662–675.

[18] L. McLinden: *The complementarity problem for maximal monotone multifunctions.* In R. W. Cottle, F. Giannessi and J. L. Lions (eds.): *Variational Inequalities and Complementarity Problems.* Academic Press, New York, 1980, pp. 251–270.

[19] J. J. Moré: *Classes of functions and feasibility conditions in nonlinear complementarity problems.* Mathematical Programming 6, 1974, pp. 327–338.

[20] J. J. Moré and W. C. Rheinboldt: *On $P-$ and $S-$functions and related classes of $n-$dimensional nonlinear mappings.* Linear Algebra and its Applications 6, 1973, pp. 45–68.

[21] J.–S. Pang: *Newton's method for B–differentiable equations.* Mathematics of Operations Research 15, 1990, pp. 311–341.

[22] J.–S. Pang and L. Qi: *Nonsmooth equations: motivation and algorithms.* SIAM Journal on Optimization 3, 1993, pp. 443–465.

[23] L. Qi and J. Sun: *A nonsmooth version of Newton's method.* Mathematical Programming 58, 1993, pp. 353–367.