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A-posteriori error estimates for optimal control problems with state and control constraints

Arnd Rösch · Daniel Wachsmuth

Abstract   We discuss the full discretization of an elliptic optimal control problem with pointwise control and state constraints. We provide the first reliable a-posteriori error estimator that contains only computable quantities for this class of problems. Moreover, we show, that the error estimator converges to zero if one has convergence of the discrete solutions to the solution of the original problem. The theory is illustrated by numerical tests.

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1 Introduction

In this paper we consider the optimal control problem of minimizing the cost functional J given by

$$J(y, u) = \frac{1}{2}\|y - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|u\|_{L^2(\Omega)}^2 \tag{1.1}$$

A. Rösch
Fakultät für Mathematik, Universität Duisburg-Essen, Forsthausweg 2, 47057 Duisburg, Germany
e-mail: arnd.roesch@uni-due.de

D. Wachsmuth (✉)
Johann Radon Institute for Computational and Applied Mathematics (RICAM),
Austrian Academy of Sciences, Altenbergerstraße 69, 4040 Linz, Austria
e-mail: daniel.wachsmuth@oeaw.ac.at

subject to

$$-\Delta y + y = u \quad \text{in } \Omega$$
$$y = 0 \quad \text{on } \Gamma,$$

(1.2)

and

$$u_a \leq u \leq u_b \quad \text{a.e. in } \Omega,$$

(1.3)

$$y_a \leq y \leq y_b \quad \text{in } \Omega.$$

(1.4)

Let us define the set of admissible controls by

$$U_{ad} := \{u \in L^2(\Omega) : u_a \leq u \leq u_b \text{ a.e. in } \Omega\}.$$

In order to guarantee existence and regularity of solutions, we assume for the whole paper:

Assumption 1 $\Omega \subset \mathbb{R}^n, n \in \{2, 3\}$, is a convex polygonal domain with boundary $\Gamma = \partial\Omega; u_a < u_b$ are constants; $y_a, y_b \in C(\bar{\Omega}), y_d \in L^2(\Omega), \alpha > 0$.

Note that our problem and also the discretized counterparts are strictly convex. Therefore, one has only to show the existence of a feasible point to get existence and uniqueness of an optimal control $\bar{u}$ and state $\bar{y}$. Under a Slater type assumption we will state such a result in Theorem 2.3. Using the same assumption, one obtains existence and uniqueness of solutions $(\bar{y}_h, \bar{u}_h)$ for sufficiently fine discretizations.

The distance of the solution $(\bar{y}_h, \bar{u}_h)$ to $(\bar{y}, \bar{u})$ can be estimated a-priori as $\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)} + \|\bar{y}_h - \bar{y}\|_{H^1(\Omega)} \leq Ch^{2-\frac{n}{2}-\varepsilon}, \varepsilon \geq 0$, see e.g. [9,19]. In order to generate adaptively refined meshes, computable and localized a-posteriori error estimators are inevitable. For work on dual-weighted residual error indicators for state constrained problems we refer to [3,11,12,24]. The main drawback of this method is that the resulting error estimators are not computable since they depend on the unknown solution of the continuous problem. The only work in context of residual error estimates is due to [16]. However, this error estimator contains also an uncomputable part that involves the Lagrange multipliers of the continuous problem. All cited papers contain different heuristics to replace uncomputable quantities by computable ones. Our main goal is to derive *computable* error bounds.

The main difficulty to obtain a computable error bound is that the errors in the dual quantities $\|\bar{p}_h - \bar{p}\|$ and $\|\bar{\mu}_{i,h} - \bar{\mu}_i\|$ cannot be majorized by $\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)}$. Hence, we completely avoid the use of the Lagrange multipliers and of the first-order necessary optimality system of Theorem 2.3. In our main result Theorem 3.14 we describe detailed all the different error contributions.

Moreover, we prove that the error estimate converges to zero if the discrete quantities converge in a certain sense, see Sect. 4. Such a result does not seem to be available even for pure control constrained problems, see the comments at the end of that section.

Let us shortly describe the structure of our paper. Section 2 contains basic properties of the optimal control problem and its discrete counterpart. A-posteriori error estimates

are derived in Sect. 3. The behavior of the error bound when the discrete solutions converge is studied in Sect. 4. Numerical experiments are presented in Sect. 5.

2 Optimality system and discretization

2.1 The undiscretized problem

First, let us define the notion of weak solutions of the state equation (1.2). A function $y \in H_0^1(\Omega)$ is called weak solution of (1.2) if it satisfies the weak formulation

$$a(y, v) = (u, v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega), \tag{2.1}$$

where the bilinear form a is defined as

$$a(y, v) = (\nabla y, \nabla v)_{L^2(\Omega)^n} + (y, v)_{L^2(\Omega)}.$$

Let us define the operators $A : V := H^2(\Omega) \cap H_0^1(\Omega) \rightarrow L^2(\Omega)$ by $A = -\Delta + I$ and its dual $A^* : L^2(\Omega) \rightarrow V^*$ by

$$(Aw, v)_{L^2(\Omega)} = \langle w, A^* v \rangle \quad \forall w \in V, \ v \in L^2(\Omega).$$

Lemma 2.1 *For each control $u \in L^2(\Omega)$ the state equation* (1.2) *admits a unique weak solution $y \in H^2(\Omega)$, and the mapping $u \mapsto y(u)$ is continuous from $L^2(\Omega)$ to $H^2(\Omega)$, i.e. $\|y\|_{H^2(\Omega)} \leq C_0 \|u\|_{L^2(\Omega)}$.*

For the proof we refer to Grisvard [10].

Throughout the article we will assume the existence of a Slater point:

Assumption 2 There exists $\hat{u} \in U_{ad}$ and $\tau \in \mathbb{R}$, $\tau > 0$, such that the associated state $\hat{y}$ satisfies $y_a + \tau \leq \hat{y} \leq y_b - \tau$.

Please note, that this assumption implies that the state constraints cannot be active on Γ, i.e. it holds $y_a < -\tau$ and $\tau < y_b$ on Γ since $\hat{y} = 0$ on Γ due to the Dirichlet boundary conditions.

Additionally, Assumption 2 implies that the feasible set of the control problem is non-empty. Due to convexity, we get immediately the existence and uniqueness of solutions.

Lemma 2.2 *Under Assumption 2, the optimal control problem* (1.1)–(1.4) *admits a unique solution $(\bar{y}, \bar{u})$.*

The solution of the optimal control problem can be characterized by means of first-order necessary optimality conditions. Due to Assumption 2, one can prove existence of Lagrange multipliers, see e.g. [6,23]. In the following, let us denote by $M(\Omega) = C(\bar{\Omega})^*$ the space of regular Borel measures.

Theorem 2.3 *Let* $(\bar{y}, \bar{u})$ *be a solution of the problem* (1.1)–(1.3). *Then there are* $\bar{\mu}_a, \bar{\mu}_b \in M(\Omega)$ *and* $\bar{p} \in W^{1,s}(\Omega), s < \frac{n}{n-1}$, *such that the following system is satisfied, which consists of adjoint equation*

$$-\Delta\bar{p} + \bar{p} = \bar{y} - y_d - \bar{\mu}_a + \bar{\mu}_b \quad in \; \Omega,$$
$$\bar{p} = 0 \quad on \; \Gamma, \tag{2.2}$$

complementarity conditions for $\bar{\mu}_a$ *and* $\bar{\mu}_b$

$$\bar{\mu}_i \geq 0, \quad \langle \bar{\mu}_i, \bar{y} - y_i \rangle = 0 \quad i \in \{a, b\}, \tag{2.3}$$

and the variational inequality

$$(\alpha\bar{u} + \bar{p}, \; u - \bar{u})_{L^2(\Omega)} \geq 0 \quad \forall u \in U_{ad}. \tag{2.4}$$

For the proof we refer to Casas [6].

2.2 Discretization

Let us fix the assumptions on the discretization of problem (1.1)–(1.4) by finite elements. First let us specify the notation of regular meshes. Each mesh $\mathcal{T}_h$ consists of closed cells T (for example triangles, tetrahedra, etc.) such that $\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} T$ holds. We assume that the mesh is regular in the following sense: for different cells $T_i, T_j \in \mathcal{T}_h, i \neq j$, the intersection $T_i \cap T_j$ is either empty or a node, an edge, or a face of both cells, i.e. hanging nodes are not allowed. Let us denote the size of each cell by $h_T = \text{diam}\, T$ and define $h(\mathcal{T}_h) = \max_{T \in \mathcal{T}_h} h_T$. For each $T \in \mathcal{T}_h$, we define R_T to be the diameter of the largest ball contained in T.

We will work with a family of regular meshes $\mathcal{F} = \{\mathcal{T}_h\}_{h>0}$, where the meshes are indexed by their mesh size, i.e. $h(\mathcal{T}_h) = h$. We assume in addition that there exists a positive constant R such that

$$\frac{h_T}{R_T} \leq R$$

holds for all cells $T \in \mathcal{T}_h$ and all $h > 0$. With each mesh $\mathcal{T}_h \in \mathcal{F}$, we associate the finite-dimensional space $V_h \subset H_0^1(\Omega)$ that consists of polynomial finite element functions of degree $l \geq 1$.

Furthermore, let us denote by $U_h \subset L^2(\Omega)$ the corresponding control discretization. Here, we have the following three possibilities in mind: discretization of controls by piecewise constant or linear finite element functions, or the choice $U_h = L^2(\Omega)$, which corresponds to the so-called variational discretization introduced by Hinze [14].

Let us now introduce the discretized version of the optimal control problem (1.1)–(1.4). This problem is given as: Find $y_h \in V_h$ and $u_h \in U_h$ that minimize

$$\min J(y_h, u_h) = \frac{1}{2}\|y_h - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|u_h\|_{L^2(\Omega)}^2 \tag{2.5}$$

subject to the discretized state equation

$$a(y_h, v_h) = (u_h, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h, \tag{2.6}$$

the control constraints

$$u_a \le u_h \le u_b \quad \text{a.e. in } \Omega, \tag{2.7}$$

and the discrete state constraints

$$y_a(x_i) \le y_h(x_i) \le y_b(x_i) \quad \text{for } i = 1, \ldots, K_h. \tag{2.8}$$

Here, we denoted by x_i, $i = 1, \ldots, K_h$, the nodes of the mesh $\mathcal{T}_h$.

In the sequel, we will rely on the following assumption:

Assumption 3 We assume that the discrete optimal control problem admits a unique solution.

Let us now discuss sufficient conditions such that this assumption is fulfilled. We first state some auxiliary results.

Lemma 2.4 *Let y^h and y_h be the solution of (2.1) and (2.6) for the control u_h. Let $C_M > 1$ be given such that $\max_{T \in \mathcal{T}_h} h_T \le C_M \min_{T \in \mathcal{T}_h} h_T$ is satisfied. Then the following L^∞-error estimate holds*

$$\|y_h - y^h\|_{L^\infty(\Omega)} \le ch\|u_h\|_{L^2(\Omega)}.$$

For a proof, we refer to Braess [4].

Let us denote by Π_h the $L^2(\Omega)$-projection onto U_h for piecewise constant functions. For piecewise linear functions one can use a quasi-interpolation, see [20]. In the case of the variational discretization we take $\Pi_h = I$. In all three cases, this operator has the following known approximation property.

Lemma 2.5 *There is a constant c_I independent of h such that*

$$\|u - \Pi_h u\|_{L^2(\Omega)} \le c_I h\|\nabla u\|_{L^2(\Omega)^n}$$

is fulfilled for all $u \in H^1(\Omega)$.

Regarding existence of solutions of the discrete optimal control problem, we have the following result.

Lemma 2.6 *Let $C_M > 1$ be given and $\max_{T \in \mathcal{T}_h} h_T \le C_M \min_{T \in \mathcal{T}_h} h_T$. Then, there exists a mesh size $h_0 > 0$ such that for all $h \le h_0$ a feasible point of the discretized problem exists.*

Proof We set $\hat{u}_h = \Pi_h \hat{u}$, where $\hat{u}$ is the Slater point from Assumption 2. The function $\hat{u}_h$ satisfies the control constraints. It remains to check the state constraints. Let us denote the solutions of the discrete and continuous state equation associated to $\hat{u}_h$ by $\hat{y}_h$ and $\hat{y}^h$, respectively. We find

$$\hat{y}_h(x_i) - y_a(x_i) \geq \tau - |\hat{y}(x_i) - \hat{y}^h(x_i)| - |\hat{y}^h(x_i) - \hat{y}_h(x_i)|$$
$$\geq \tau - \|\hat{y} - \hat{y}^h\|_{L^\infty(\Omega)} - \|\hat{y}^h - \hat{y}_h\|_{L^\infty(\Omega)}$$

Here, $\|\hat{y} - \hat{y}^h\|_{L^\infty(\Omega)}$ becomes small for small h because of Lemmas 2.5 and 2.1. The term $\|\hat{y}^h - \hat{y}_h\|_{L^\infty(\Omega)}$ reflects the pointwise discretization error, which tends to zero for $h \to 0$ due to Lemma 2.4. Consequently, for sufficiently small h the discrete lower state constraints are fulfilled. Analogously, one shows that the upper state constraint of the discrete problem is satisfied for h small enough. Hence, the point $(\hat{y}_h, \hat{u}_h)$ is admissible for the discrete problem.   □

As for the continuous problem, we get existence and uniqueness of solutions of the discrete problem.

Lemma 2.7 *Let Assumption 2 be satisfied and $\max_{T \in \mathcal{T}_h} h_T \leq C_M \min_{T \in \mathcal{T}_h} h_T$. Let the mesh-size satisfy $h < h_0$. Then the optimal control problem (2.5)–(2.8) admits a uniquely determined solution $(\bar{y}_h, \bar{u}_h)$ for all meshes with mesh-size $h < h_0$.*

Proof Due to Lemma 2.6 the feasible set of the discrete problem is non-empty for $h < h_0$. By standard arguments one concludes the existence of a unique solution of this problem.   □

Remark 2.8 Let us note that the existence of a feasible point for the discrete problem is not guaranteed for arbitrary meshes with $h < h_0$. This is due to the fact that a-priori L^∞-error estimates are derived using inverse inequalities. To apply standard results, the relation between minimal and maximal element size must be limited by some constant C_M. Another possibility are meshes for which a discrete maximum principle holds, see [8]. Usually, sequences of adaptive refined grids do not satisfy these requirements. Here it is an open question, how the existence of feasible points for the discrete problem can be maintained or regained during adaptive refinement.

Analogous to the continuous problem, one finds that the solution of the discrete problem can be characterized by a first-order optimality system.

Theorem 2.9 *Let $(\bar{y}_h, \bar{u}_h)$ be a solution of the problem (2.5)–(2.8). Then there are $\bar{\mu}_{a,h}, \bar{\mu}_{b,h} \in M(\Omega)$ and $\bar{p}_h \in V_h$, such that the following system is satisfied, which consists of discrete adjoint equation*

$$a(v_h, \bar{p}_h) = (\bar{y}_h - y_d, v_h)_{L^2(\Omega)} + \langle -\bar{\mu}_{a,h} + \bar{\mu}_{b,h}, v_h \rangle \quad \forall v_h \in V_h, \qquad (2.9)$$

complementarity condition

$$\bar{\mu}_{i,h} = \sum_{j=1}^{K_h} \bar{\mu}_{i,h}^j \delta(x_j), \ \bar{\mu}_{i,h}^j \geq 0,$$

$$\bar{\mu}_{i,h}^j(\bar{y}_h(x_j) - y_i(x_j)) = 0, \quad i \in \{a, b\}, j \in \{1, \ldots, K_h\}, \quad (2.10)$$

and variational inequality

$$(\alpha \bar{u}_h + \bar{p}_h, \ u_h - \bar{u}_h)_{L^2(\Omega)} \geq 0 \ \ \forall u_h \in U_h \cap U_{ad}.$$

Since the state constraints for the discrete problem were prescribed in the mesh nodes, the Lagrange multipliers $\bar{\mu}_{a,h}$ and $\bar{\mu}_{b,h}$ are positive linear combinations of Dirac measures concentrated in the mesh nodes. Due to this representation, the complementarity condition in (2.10) can be written as

$$\langle \bar{\mu}_{i,h}, \ \bar{y}_h - y_i \rangle = 0 \ \ i \in \{a, b\}.$$

3 A-posteriori error estimates

In this section, we will derive an upper bound for the error $\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)}$. In order to avoid the difficulties mentioned in the introduction, we will not work with the first-order optimality system given by Theorem 2.3. We will work with the optimality of $\bar{u}$ instead, i.e. $J(\bar{y}, \bar{u}) \leq J(y, u)$ for all (y, u) satisfying the constraints (1.2)–(1.4).

Let us suppose that we have computed the solution $(\bar{y}_h, \bar{u}_h, \bar{p}_h, \bar{\mu}_{a,h}, \bar{\mu}_{b,h})$ of the discrete problem. At first, we have to find a pair $(\tilde{y}, \tilde{u}) \approx (\bar{y}_h, \bar{u}_h)$ that is feasible for the continuous problem, which would give $J(\bar{y}, \bar{u}) \leq J(\tilde{y}, \tilde{u})$.

3.1 Construction of feasible control

Let us first define an auxiliary state y^h as the weak solution of the state equation with right-hand side $\bar{u}_h$, i.e.

$$a(y^h, v) = (\bar{u}_h, v)_{L^2(\Omega)} \ \ \forall v \in H_0^1(\Omega). \tag{3.1}$$

Lemma 3.1 *Let $\hat{u}$ denote the Slater point as given by Assumption 2. Let y^h denote the solution of (3.1). Let us define the violation of the state constraints by*

$$e_{sc} := \max(\|(\bar{y}_h - y_a)^-\|_{L^\infty(\Omega)}, \ \|(y_b - \bar{y}_h)^-\|_{L^\infty(\Omega)}). \tag{3.2}$$

Then the state $\tilde{y} := (1 - \sigma)y^h + \sigma \hat{y}$ associated to the control $\tilde{u} := (1 - \sigma)\bar{u}_h + \sigma \hat{u}$ is admissible for the state constraints (1.4) if σ is chosen as

$$\sigma = \frac{\|\bar{y}_h - y^h\|_{L^\infty(\Omega)} + e_{sc}}{\tau + \|\bar{y}_h - y^h\|_{L^\infty(\Omega)} + e_{sc}}.$$

Proof We find

$$
\begin{aligned}
(1-\sigma)y^h + \sigma\hat{y} &\geq (1-\sigma)(\bar{y}_h - \|\bar{y}_h - y^h\|_{L^\infty(\Omega)}) + \sigma(y_a + \tau) \\
&\geq y_a + \tau + (1-\sigma)(\bar{y}_h - \|\bar{y}_h - y^h\|_{L^\infty(\Omega)} - y_a - \tau) \\
&\geq y_a + \tau + (1-\sigma)(-\|(\bar{y}_h - y_a)^-\|_{L^\infty(\Omega)} - \|\bar{y}_h - y^h\|_{L^\infty(\Omega)} - \tau) \\
&\geq y_a + \tau - (1-\sigma)(\tau + \|\bar{y}_h - y^h\|_{L^\infty(\Omega)} + e_{sc}),
\end{aligned}
$$

which implies $(1-\sigma)y^h + \sigma\hat{y} \geq y_a$ for $1-\sigma \leq \frac{\tau}{\tau + \|\bar{y}_h - y^h\|_{L^\infty(\Omega)} + e_{sc}}$. An analogous discussion for the upper state constraint yields the claim. □

With the notation of the previous lemma, we have

$$
\tilde{u} - \bar{u}_h = \sigma(\hat{u} - \bar{u}_h), \tag{3.3}
$$

which allows to estimate the difference $\|\tilde{u} - \bar{u}_h\|_{L^2(\Omega)}$ provided upper bounds for σ are available. The difference in the states $\tilde{y} - \bar{y}_h$ can be written as

$$
\begin{aligned}
\tilde{y} - \bar{y}_h = \tilde{y} - y^h + y^h - \bar{y}_h &= \sigma(\hat{y} - y^h) + y^h - \bar{y}_h \\
&= \sigma(\hat{y} - \bar{y}_h) + (1-\sigma)(y^h - \bar{y}_h). \tag{3.4}
\end{aligned}
$$

Remark 3.2 Please note, that we did not use feasibility of $\bar{y}_h$ for the discrete optimization problem. Hence, Lemma 3.1 is valid without this assumption, which means it is also applicable if the discrete optimization problem is solved for instance by penalty methods.

Remark 3.3 If the discrete space V_h is the space of piecewise linear polynomial functions, then we can replace the constraint violation e_{sc} by the interpolation error of the state constraint bounds. Let I_h denote the Lagrange (or nodal) interpolation operator. Then the discrete state constraints (2.8) imply $I_h y_a \leq \bar{y}_h \leq I_h y_b$ on $\bar{\Omega}$. And we can estimate in the previous proof

$$
\begin{aligned}
(1-\sigma)y^h + \sigma\hat{y} &\geq (1-\sigma)(\bar{y}_h - \|\bar{y}_h - y^h\|_{L^\infty(\Omega)}) + \sigma(y_a + \tau) \\
&\geq y_a + \tau + (1-\sigma)(I_h y_a - \|\bar{y}_h - y^h\|_{L^\infty(\Omega)} - y_a - \tau),
\end{aligned}
$$

which shows that we can replace e_{sc} by

$$
\tilde{e}_{sc} := \max(\|(I_h y_a - y_a)^-\|_{L^\infty(\Omega)}, \ \|(y_b - I_h y_b)^-\|_{L^\infty(\Omega)}).
$$

If the functions y_a, y_b can be exactly represented by piecewise linear functions then $\tilde{e}_{sc} = 0$. This is in particular the case if both state constraints are constant functions.

Remark 3.4 The result of Lemma 3.1 remains valid, if one replaces e_{sc} by

$$
\hat{e}_{sc} := \max(\text{ess}\sup(y_a - y_h), \ \text{ess}\sup(y_h - y_b)).
$$

If y_h is strictly between the state constraints, then $\hat{e}_{\mathrm{sc}}$ is negative. Hence, if $\hat{e}_{\mathrm{sc}} \leq -\|y_h - y^h\|_{L^\infty(\Omega)}$ then one can choose $\sigma = 0$. This allows to omit the $L^\infty(\Omega)$-error of the states in the case of inactive state constraints or if no state constraints are prescribed.

Remark 3.5 The application of Lemma 3.1 requires the explicit knowledge of the Slater point $(\hat{y}, \hat{u})$. For simple problems such a point can be determined by hand. If the underlying elliptic equation does not permit this, one can use the following strategy. First, one computes a discrete Slater point $(\hat{y}_h, \hat{u}_h)$ satisfying the discrete state equation, the control constraints, and strict discrete state constraints $y_a(x_i) + \tau_h \leq \hat{y}_h(x_i) \leq y_b(x_i) - \tau_h, i = 1, \ldots, K_h$, with some fixed $\tau_h > 0$. Secondly, one has to compute constant-free error estimates of the type $\|\hat{y}_h - \hat{y}^h\|_{L^\infty(\Omega)} \leq \hat{\varepsilon}$, where $\hat{y}^h$ is the solution of the state equation associated to $\hat{u}_h$. This can be achieved using a-posteriori enclosure methods, see Plum [22, Section 4] for a detailed description. If $\hat{\varepsilon} < \tau_h$ holds then $(\hat{y}^h, \hat{u}_h)$ is the wanted Slater point, where the function $\hat{y}^h$ remains unknown, but the distance to y_h can be bounded by $\|y_h - \hat{y}^h\|_{L^\infty(\Omega)} \leq \|y_h - \hat{y}_h\|_{L^\infty(\Omega)} + \hat{\varepsilon}$.

However, if one is willing to accept the assumption of existence of an unknown Slater point, then the estimate of Lemma 3.1 can be transformed into $\sigma \leq c \, (\|y^h - y_h\|_{L^\infty(\Omega)} + e_{\mathrm{sc}})$ with a positive constant $c > 0$. Proceeding this way, the Slater point $(\hat{y}, \hat{u})$ as well as the distance τ are hidden in some uniformly bounded constant in the resulting error estimator.

3.2 Estimate of error in the control variational inequality

As one ingredient of the final error estimator we will develop an error estimator for the error in the variational inequality (2.4). We will comment on the relation to existing work at the end of Sect. 4.

At first, let us define the following subsets of Ω

$$\Omega_{0,h} := \left\{ x \in \Omega : \ \bar{u}_h(x), -\frac{1}{\alpha}\bar{p}_h(x) \in (u_a, u_b) \right.$$
$$\text{or } \bar{u}_h(x) = u_a, \alpha\bar{u}_h(x) + \bar{p}_h(x) < 0$$
$$\left. \text{or } \bar{u}_h(x) = u_b, \alpha\bar{u}_h(x) + \bar{p}_h(x) > 0 \right\} \tag{3.5}$$

and

$$\Omega_{a,h} := \left\{ x \in \Omega : \bar{u}_h(x) \in (u_a, u_b), -\frac{1}{\alpha}\bar{p}_h(x) \leq u_a \right\}$$
$$\Omega_{b,h} := \left\{ x \in \Omega : \bar{u}_h(x) \in (u_a, u_b), -\frac{1}{\alpha}\bar{p}_h(x) \geq u_b \right\}. \tag{3.6}$$

The set $\Omega_{0,h}$ contains the points, where $\bar{u}_h$ and $-\frac{1}{\alpha}\bar{p}_h$ are strict between the bounds, and where $\bar{u}_h$ is at the bound but $\alpha\bar{u}_h + \bar{p}_h$ has the wrong sign. The sets $\Omega_{a,h}$ and $\Omega_{b,h}$ contain points, where $\bar{u}_h$ is strictly between the bounds, but $-\frac{1}{\alpha}\bar{p}_h$ is not feasible with

respect to these bounds. In addition, we have the following properties

$$\alpha \bar{u}_h + \bar{p}_h \geq \alpha(\bar{u}_h - u_a) > 0 \text{ on } \Omega_{a,h},$$
$$\alpha \bar{u}_h + \bar{p}_h \leq \alpha(\bar{u}_h - u_b) < 0 \text{ on } \Omega_{b,h}. \tag{3.7}$$

Lemma 3.6 *Let $\bar{u}_h \in U_{ad}$ and $\bar{p}_h \in L^2(\Omega)$ be given. Let the sets $\Omega_{0,h}, \Omega_{a,h},$ and $\Omega_{b,h}$ be defined according to (3.5) and (3.6). Then for each $u \in U_{ad}$ it holds*

$$(\alpha \bar{u}_h + \bar{p}_h, u - \bar{u}_h) \geq (\chi_{\Omega_{0,h}}(\alpha \bar{u}_h + \bar{p}_h), u - \bar{u}_h)$$
$$+ (\chi_{\Omega_{a,h}}(\alpha \bar{u}_h + \bar{p}_h), u_a - \bar{u}_h) + (\chi_{\Omega_{b,h}}(\alpha \bar{u}_h + \bar{p}_h), u_b - \bar{u}_h)$$

Proof The proof follows directly from the definition of the sets, and the properties of $\bar{u}_h$ and $\alpha \bar{u}_h + \bar{p}_h$ on these sets, confer also (3.7). $\qquad\square$

In the derivation of the a-posteriori error estimate in Sect. 3.4, we will use the following implication of the previous lemma.

Lemma 3.7 *Let $\bar{u}_h \in U_{ad}$ and $\bar{p}_h \in L^2(\Omega)$ be given. Let the sets $\Omega_{0,h}, \Omega_{a,h},$ and $\Omega_{b,h}$ be defined according to (3.5) and (3.6). Then for each $u \in U_{ad}$ it holds*

$$(\alpha \bar{u}_h + \bar{p}_h, u - \bar{u}_h) \geq -\frac{\alpha}{4} \|u - \bar{u}_h\|_{L^2(\Omega)}^2 - \|\eta_{vi}\|_{L^2(\Omega)}^2,$$

where $\eta_{vi} = \eta_{vi}(\bar{u}_h, \bar{p}_h) \in L^2(\Omega)$ is given by

$$\eta_{vi}^2 = \frac{1}{\alpha} \chi_{\Omega_{0,h}}(\alpha \bar{u}_h + \bar{p}_h)^2 + \chi_{\Omega_{a,h}}(\alpha \bar{u}_h + \bar{p}_h)(\bar{u}_h - u_a)$$
$$+ \chi_{\Omega_{b,h}}(\alpha \bar{u}_h + \bar{p}_h)(\bar{u}_h - u_b). \tag{3.8}$$

Proof The claim follows directly from the definition of η_{vi} and the inequality

$$(\chi_{\Omega_{0,h}}(\alpha \bar{u}_h + \bar{p}_h), u - \bar{u}_h) \geq -\frac{\alpha}{4} \|u - \bar{u}_h\|_{L^2(\Omega)}^2 - \frac{1}{\alpha} \|\alpha \bar{u}_h + \bar{p}_h\|_{L^2(\Omega_{0,h})}^2.$$

$\qquad\square$

The function η_{vi} will serve as a localizable error estimator for the error in the variational inequality.

Remark 3.8 Let us comment on the situation when the variational discretization is applied. Then by construction of this discretization concept it holds $(\alpha \bar{u}_h + \bar{p}_h, u - \bar{u}_h) \geq 0$ for all $u \in U_{ad}$ if $(\bar{u}_h, \bar{p}_h)$ are the optimal discrete control and adjoint. Then $\eta_{vi} = 0$ in this case.

Remark 3.9 If no control constraints are prescribed, then it holds $\eta_{vi}^2 = \frac{1}{\alpha}(\alpha u_h + p_h)^2$. If $(\bar{u}_h, \bar{p}_h)$ are the optimal discrete control and adjoint, and $V_h \subset U_h$ then it holds $\eta_{vi} = 0$.

3.3 Estimate of cost functional values with respect to the admissible point

Due to the feasibility of $(\tilde{y}, \tilde{u})$ the inequality $0 \leq J(\tilde{y}, \tilde{u}) - J(\bar{y}, \bar{u})$ holds. Now, we will derive an upper bound for $J(\tilde{y}, \tilde{u}) - J(y, u)$ for arbitrary feasible pairs (y, u) in terms of the distance $\|u - \tilde{u}\|_{L^2(\Omega)}$ and of residuals of the optimality system.

Lemma 3.10 *Let $(\tilde{y}, \tilde{u})$ be given by Lemma* 3.1. *Then it holds for all (y, u) satisfying* (1.2)–(1.4)

$$J(\tilde{y}, \tilde{u}) - J(y, u) \leq -\frac{\alpha}{4}\|u - \tilde{u}\|^2_{L^2(\Omega)} + \eta_a\|u - \tilde{u}\|_{L^2(\Omega)} + \eta_b,$$

where η_a, η_b are real numbers depending on $(\bar{y}_h, \bar{u}_h, \bar{p}_h, \bar{\mu}_{a,h}, \bar{\mu}_{b,h})$, $(\tilde{y}, \tilde{u})$, and the data of the problem but not on (y, u), see below (3.13).

Proof Let us introduce the abbreviation $r := \|u - \tilde{u}\|_{L^2(\Omega)}$.

Since $\tilde{y}$ and y are solutions to the elliptic equations for right-hand sides $\tilde{u}$ and u respectively, we have by Lemma 2.1 the regularity $\tilde{y}, y \in H^2(\Omega) \cap H^1_0(\Omega)$. Since we have $\bar{p}_h \in V_h \subset L^2(\Omega)$ the dual product $\langle A\tilde{y} - Ay, \bar{p}_h \rangle$ is well-defined.

Let us write the differences of the cost functional as

$$\begin{aligned}
J(\tilde{y}, \tilde{u}) - J(y, u) &= \frac{1}{2}\|\tilde{y} - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\tilde{u}\|^2_{L^2(\Omega)} - \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} - \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \\
&\quad + \langle A\tilde{y} - \tilde{u} - Ay + u, \bar{p}_h \rangle \\
&= -\frac{1}{2}\|\tilde{y} - y\|^2_{L^2(\Omega)} - \frac{\alpha}{2}\|\tilde{u} - u\|^2_{L^2(\Omega)} + (\alpha\tilde{u} + \bar{p}_h, \tilde{u} - u) \\
&\quad + \langle -A^*\bar{p}_h + \bar{y}_h - y_d, \tilde{y} - y \rangle + (\tilde{y} - \bar{y}_h, \tilde{y} - y) \\
&\leq -\frac{\alpha}{2}r^2 + \frac{1}{2}\|\tilde{y} - \bar{y}_h\|^2_{L^2(\Omega)} + (\alpha\tilde{u} + \bar{p}_h, \tilde{u} - u) \\
&\quad + \langle -A^*\bar{p}_h + \bar{y}_h - y_d, \tilde{y} - y \rangle.
\end{aligned} \tag{3.9}$$

Now we will estimate the third and fourth addend on the right-hand side. By Lemma 3.7 there exists $\eta_{\mathrm{vi}} \in L^2(\Omega)$ such that

$$(\alpha\bar{u}_h + \bar{p}_h, u - \bar{u}_h) \geq -\frac{\alpha}{4}\|u - \bar{u}_h\|^2_{L^2(\Omega)} - \|\eta_{\mathrm{vi}}\|^2_{L^2(\Omega)} \quad \forall u \in L^2(\Omega) : u_a \leq u \leq u_b.$$

We then obtain

$$\begin{aligned}
(\alpha\tilde{u} + \bar{p}_h, \tilde{u} - u) &= \alpha(\tilde{u} - \bar{u}_h, \tilde{u} - u) + (\alpha\bar{u}_h + \bar{p}_h, \tilde{u} - \bar{u}_h) + (\alpha\bar{u}_h + \bar{p}_h, \bar{u}_h - u) \\
&\leq \alpha(\tilde{u} - \bar{u}_h, \tilde{u} - u) + (\alpha\bar{u}_h + \bar{p}_h, \tilde{u} - \bar{u}_h) \\
&\quad + \frac{\alpha}{4}\|u - \bar{u}_h\|^2_{L^2(\Omega)} + \|\eta_{\mathrm{vi}}\|^2_{L^2(\Omega)} \\
&\leq \frac{\alpha}{4}r^2 + \frac{3}{2}\alpha\|\tilde{u} - \bar{u}_h\|_{L^2(\Omega)}\, r + \frac{\alpha}{4}\|\tilde{u} - \bar{u}_h\|^2_{L^2(\Omega)} \\
&\quad + (\alpha\bar{u}_h + \bar{p}_h, \tilde{u} - \bar{u}_h) + \|\eta_{\mathrm{vi}}\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.10}$$

Due to the complementarity condition (2.10) on $\bar{\mu}_{a,h}$ and the feasibility $y_a \leq y$ it holds

$$
\begin{aligned}
\langle \bar{\mu}_{a,h}, \, \tilde{y} - y \rangle &= \langle \bar{\mu}_{a,h}, \, \tilde{y} - \bar{y}_h + \bar{y}_h - y_a + y_a - y \rangle \\
&= \langle \bar{\mu}_{a,h}, \, \tilde{y} - \bar{y}_h \rangle + \langle \bar{\mu}_{a,h}, \, y_a - y \rangle \\
&\leq \langle \bar{\mu}_{a,h}, \, \tilde{y} - \bar{y}_h \rangle.
\end{aligned}
$$

For analogous reasons we find

$$
-\langle \bar{\mu}_{b,h}, \, \tilde{y} - y \rangle \leq -\langle \bar{\mu}_{b,h}, \, \tilde{y} - \bar{y}_h \rangle.
$$

Thus, the fourth addend in (3.9) can be estimated as

$$
\begin{aligned}
&\langle -A^* \bar{p}_h + \bar{y}_h - y_d, \, \tilde{y} - y \rangle \\
&= \langle -A^* \bar{p}_h + \bar{y}_h - y_d - \bar{\mu}_{a,h} + \bar{\mu}_{b,h}, \, \tilde{y} - y \rangle + \langle \bar{\mu}_{a,h}, \, \tilde{y} - y \rangle - \langle \bar{\mu}_{b,h}, \, \tilde{y} - y \rangle \\
&\leq \langle -A^* \bar{p}_h + \bar{y}_h - y_d - \bar{\mu}_{a,h} + \bar{\mu}_{b,h}, \, \tilde{y} - y \rangle + \langle \bar{\mu}_{a,h} - \bar{\mu}_{b,h}, \, \tilde{y} - \bar{y}_h \rangle \\
&\leq C_0 \| -A^* \bar{p}_h + \bar{y}_h - y_d - \bar{\mu}_{a,h} + \bar{\mu}_{b,h} \|_{H^{-2}(\Omega)} r + \langle \bar{\mu}_{a,h} - \bar{\mu}_{b,h}, \, \tilde{y} - \bar{y}_h \rangle,
\end{aligned}
\tag{3.11}
$$

where we applied the $H^2(\Omega)$-regularity result Lemma 2.1 in the last step.

Combining (3.9)–(3.11), we can estimate the difference of the values of the cost functional as

$$
\begin{aligned}
J(\tilde{y}, \tilde{u}) - J(y, u) \leq &-\frac{\alpha}{4} r^2 \\
&+ \left(\frac{3}{2} \alpha \| \tilde{u} - \bar{u}_h \|_{L^2(\Omega)} + C_0 \| -A^* \bar{p}_h + \bar{y}_h - y_d - \bar{\mu}_{a,h} + \bar{\mu}_{b,h} \|_{H^{-2}(\Omega)} \right) r \\
&+ \frac{1}{2} \| \tilde{y} - \bar{y}_h \|_{L^2(\Omega)}^2 + \frac{\alpha}{4} \| \tilde{u} - \bar{u}_h \|_{L^2(\Omega)}^2 + (\alpha \bar{u}_h + \bar{p}_h, \, \tilde{u} - \bar{u}_h) \\
&+ \| \eta_{\mathrm{vi}} \|_{L^2(\Omega)}^2 + \langle \bar{\mu}_{a,h} - \bar{\mu}_{b,h}, \, \tilde{y} - \bar{y}_h \rangle,
\end{aligned}
\tag{3.12}
$$

which yields the claim with

$$
\begin{aligned}
\eta_a &:= \frac{3}{2} \alpha \| \tilde{u} - \bar{u}_h \|_{L^2(\Omega)} + C_0 \| -A^* \bar{p}_h + \bar{y}_h - y_d - \bar{\mu}_{a,h} + \bar{\mu}_{b,h} \|_{H^{-2}(\Omega)} \\
\eta_b &:= \frac{1}{2} \| \tilde{y} - \bar{y}_h \|_{L^2(\Omega)}^2 + \frac{\alpha}{4} \| \tilde{u} - \bar{u}_h \|_{L^2(\Omega)}^2 + (\alpha \bar{u}_h + \bar{p}_h, \, \tilde{u} - \bar{u}_h) \\
&\quad + \| \eta_{\mathrm{vi}} \|_{L^2(\Omega)}^2 + \langle \bar{\mu}_{a,h} - \bar{\mu}_{b,h}, \, \tilde{y} - \bar{y}_h \rangle.
\end{aligned}
\tag{3.13}
$$

$\square$

Remark 3.11 If one uses interior point methods to solve the discretized problem, then the discrete complementarity condition (2.10), in particular $\langle \bar{\mu}_{i,h}, \, \bar{y}_h - y_i \rangle = 0, i \in$

$\{a, b\}$, is not satisfied in general. It turns out that the estimate of the previous lemma holds true if η_b is replaced by $\tilde{\eta}_b$ given by

$$\tilde{\eta}_b := \eta_b + \langle \bar{\mu}_{a,h}, \bar{y}_h - y_a \rangle - \langle \bar{\mu}_{b,h}, \bar{y}_h - y_b \rangle,$$

which takes the violation of the discrete complementarity condition into account.

Remark 3.12 The proof of Lemma 3.10 relies on the $H^2(\Omega)$-regularity of the elliptic equation. If the mapping $u \mapsto y(u)$ is continuous from $L^2(\Omega)$ to $W^{1,p}(\Omega)$, $p > n$, instead, then the result of the Lemma has to be modified in the following way. The duality pairing appearing in (3.11) has to estimated by

$$\begin{aligned}
&\langle -A^* \bar{p}_h + \bar{y}_h - y_d - \bar{\mu}_{a,h} + \bar{\mu}_{b,h}, \tilde{y} - y \rangle \\
&\quad \leq \| -A^* \bar{p}_h + \bar{y}_h - y_d - \bar{\mu}_{a,h} + \bar{\mu}_{b,h} \|_{W^{-1,p'}(\Omega)} \| \tilde{y} - y \|_{W^{1,p}(\Omega)} \\
&\quad \leq \tilde{C}_0 \| -A^* \bar{p}_h + \bar{y}_h - y_d - \bar{\mu}_{a,h} + \bar{\mu}_{b,h} \|_{W^{-1,p'}(\Omega)} \| \tilde{u} - u \|_{W^{1,p}(\Omega)},
\end{aligned}$$

with $1/p + 1/p' = 1$. In this way, all the following results stay valid if the residual in the adjoint equation is measured in the $W^{-1,p'}(\Omega)$-norm instead of the $H^{-2}(\Omega)$-norm.

3.4 Upper bound for the error in the control and state

Using the solution of the continuous problem as test functions in Lemma 3.10, we get directly an estimate of the error in the controls.

Lemma 3.13 *With the notations of the previous Lemma 3.10, it holds*

$$\| \bar{u}_h - \bar{u} \|_{L^2(\Omega)}^2 \leq 2 \| \tilde{u} - \bar{u}_h \|_{L^2(\Omega)}^2 + \frac{32}{\alpha^2} \eta_a^2 + \frac{16}{\alpha} \eta_b.$$

Proof By Lemma 3.10, optimality of $(\bar{y}, \bar{u})$, and feasibility of $(\tilde{y}, \tilde{u})$, we obtain

$$0 \leq J(\tilde{y}, \tilde{u}) - J(y, u) \leq -\frac{\alpha}{4} \| \bar{u} - \tilde{u} \|_{L^2(\Omega)}^2 + \eta_a \| \bar{u} - \tilde{u} \|_{L^2(\Omega)} + \eta_b,$$

which gives directly

$$\| \bar{u} - \tilde{u} \|_{L^2(\Omega)}^2 \leq \frac{16}{\alpha^2} \eta_a^2 + \frac{8}{\alpha} \eta_b.$$

The claim follows with $\| \bar{u}_h - \bar{u} \|_{L^2(\Omega)}^2 \leq 2(\| \tilde{u} - \bar{u}_h \|_{L^2(\Omega)}^2 + \| \bar{u} - \tilde{u} \|_{L^2(\Omega)}^2)$. □

Here, the quantity η_a^2 can be bounded from above, cf. (3.13), by

$$\eta_a^2 \leq \frac{9}{2} \alpha^2 \| \tilde{u} - \bar{u}_h \|_{L^2(\Omega)}^2 + 2 C_0^2 \| -A^* \bar{p}_h + \bar{y}_h - y_d - \bar{\mu}_{a,h} + \bar{\mu}_{b,h} \|_{H^{-2}(\Omega)}^2.$$

$$\tag{3.14}$$

Now we have everything at hand to derive the upper bound for the discretization error of control and state. Let us emphasize that all quantities on the right-hand side of the error estimate (3.15) are computable. In particular, the right-hand side *does not* contain any component of the solution $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_a, \bar{\mu}_b)$ of the continuous optimality system.

Theorem 3.14 *Let $(\bar{y}, \bar{u})$ be the solution of the continuous optimal control problem (1.1)–(1.4). Let $(\bar{y}_h, \bar{u}_h, \bar{p}_h, \bar{\mu}_{a,h}, \bar{\mu}_{b,h})$ be the solution of the discrete problem satisfying (2.7)–(2.8), (2.10).*

Then there is a constant $c > 0$ that depends only on Ω, A, τ such that

$$\|\bar{u}_h - \bar{u}\|^2_{L^2(\Omega)} + \|\bar{y}_h - \bar{y}\|^2_{L^\infty(\Omega)} \leq c\,\alpha^{-1}\Big(r^{(1)}_{\text{state},L^\infty} + r^{(2)}_{\text{state},L^\infty} + r^2_{\text{state},L^2}$$
$$+ \alpha^{-1}\, r^2_{\text{adjoint},L^2} + r^2_{\text{control},L^2} + e_{\text{state}}\Big) \quad (3.15)$$

where $r^{(1)}_{\text{state},L^\infty}$ is the scaled L^∞-error of the states given by

$$r^{(1)}_{\text{state},L^\infty} = \Big(\max\big((\alpha\bar{u}_h + \bar{p}_h, \hat{u} - \bar{u}_h) + \langle\bar{\mu}_{a,h} - \bar{\mu}_{b,h}, \hat{y} - \bar{y}_h\rangle,\ 0\big)$$
$$+ \|\bar{\mu}_{a,h} - \bar{\mu}_{b,h}\|_{M(\Omega)}\Big)\|y^h - \bar{y}_h\|_{L^\infty(\Omega)},$$

$r^{(2)}_{\text{state},L^\infty}$ is the squared and scaled L^∞-error of the states given by

$$r^{(2)}_{\text{state},L^\infty} = \Big(\alpha\|\hat{u} - \bar{u}_h\|^2_{L^2(\Omega)} + \|\hat{y} - \bar{y}_h\|^2_{L^2(\Omega)}\Big)\|y^h - \bar{y}_h\|^2_{L^\infty(\Omega)},$$

r_{state,L^2} is the L^2-error of the states given by

$$r_{\text{state},L^2} = \|y^h - \bar{y}_h\|_{L^2(\Omega)},$$

r_{adjoint,L^2} is the $H^{-2}(\Omega)$-residual in the adjoint equation

$$r_{\text{adjoint},L^2} = \|-A^*\bar{p}_h + \bar{y}_h - y_d - \bar{\mu}_{a,h} + \bar{\mu}_{b,h}\|_{H^{-2}(\Omega)},$$

r_{control,L^2} is the $L^2(\Omega)$-residual in the variational inequality defined by

$$r_{\text{control},L^2} = \|\eta_{\text{vi}}\|_{L^2(\Omega)}, \text{ where } \eta_{\text{vi}} = \eta_{\text{vi}}(\bar{u}_h, \bar{p}_h)\text{is given by Lemma 3.7,}$$

and e_{state} is the weighted violation of the state constraints given by

$$e_{\text{state}} = \max\big((\alpha\bar{u}_h + \bar{p}_h, \hat{u} - \bar{u}_h) + \langle\bar{\mu}_{a,h} - \bar{\mu}_{b,h}, \hat{y} - \bar{y}_h\rangle,\ 0\big)e_{\text{sc}}$$
$$+ \Big(\alpha\|\hat{u} - \bar{u}_h\|^2_{L^2(\Omega)} + \|\hat{y} - \bar{y}_h\|^2_{L^2(\Omega)}\Big)e^2_{\text{sc}},$$

where e_{sc} is defined in Lemma 3.1, Eq. (3.2).

Proof Let us first combine the results of Lemma 3.10 with the estimate of η_a^2 in (3.14) and the definition of η_b in Lemma 3.13, Eq. (3.13), to obtain

$$
\begin{aligned}
\|\bar{u}_h - \bar{u}\|^2_{L^2(\Omega)} \leq\ & 150\|\tilde{u} - \bar{u}_h\|^2_{L^2(\Omega)} \\
& + \frac{64 C_0^2}{\alpha^2}\|-A^*\bar{p}_h + \bar{y}_h - y_d - \bar{\mu}_{a,h} + \bar{\mu}_{b,h}\|^2_{H^{-2}(\Omega)} \\
& + \frac{16}{\alpha}\left(\frac{1}{2}\|\tilde{y} - \bar{y}_h\|^2_{L^2(\Omega)} + (\alpha\bar{u}_h + \bar{p}_h, \tilde{u} - \bar{u}_h) + \|\eta_{\mathrm{vi}}\|^2_{L^2(\Omega)} \right. \\
& \left. + \langle\bar{\mu}_{a,h} - \bar{\mu}_{b,h},\ \tilde{y} - \bar{y}_h\rangle\right).
\end{aligned}
$$

At first, we have by Lemma 3.1, cf. (3.3),

$$(\alpha\bar{u}_h + \bar{p}_h, \tilde{u} - \bar{u}_h) = (\alpha\bar{u}_h + \bar{p}_h, \hat{u} - \bar{u}_h)\sigma$$

and

$$\|\tilde{u} - \bar{u}_h\|^2_{L^2(\Omega)} = \sigma^2\|\hat{u} - \bar{u}_h\|^2_{L^2(\Omega)}.$$

With the help of (3.4) we get

$$
\begin{aligned}
\langle\bar{\mu}_{a,h} - \bar{\mu}_{b,h},\ \tilde{y} - \bar{y}_h\rangle = & \ \sigma\langle\bar{\mu}_{a,h} - \bar{\mu}_{b,h},\ \hat{y} - \bar{y}_h\rangle + (1-\sigma)\langle\bar{\mu}_{a,h} - \bar{\mu}_{b,h},\ y^h - \bar{y}_h\rangle \\
\leq & \ \sigma\langle\bar{\mu}_{a,h} - \bar{\mu}_{b,h},\ \hat{y} - \bar{y}_h\rangle \\
& + \|\bar{\mu}_{a,h} - \bar{\mu}_{b,h}\|_{M(\Omega)}\|y^h - \bar{y}_h\|_{L^\infty(\Omega)}.
\end{aligned}
$$

Similarly, we can estimate using $\sigma \geq 0$

$$
\begin{aligned}
\|\tilde{y} - \bar{y}_h\|^2_{L^2(\Omega)} = & \ \|\sigma(\hat{y} - \bar{y}_h) + (1-\sigma)(y^h - \bar{y}_h)\|^2_{L^2(\Omega)} \\
\leq & \ 2\left(\sigma^2\|\hat{y} - \bar{y}_h\|^2_{L^2(\Omega)} + \|y^h - \bar{y}_h\|^2_{L^2(\Omega)}\right).
\end{aligned}
$$

Hence there is a constant $c > 0$ depending only on Ω, A such that

$$
\begin{aligned}
\|\bar{u}_h - \bar{u}\|^2_{L^2(\Omega)} \leq c\alpha^{-1}\Big\{ & \alpha\|\hat{u} - \bar{u}_h\|^2_{L^2(\Omega)}\sigma^2 \\
& + \alpha^{-1}\|-A^*\bar{p}_h + \bar{y}_h - y_d - \bar{\mu}_{a,h} + \bar{\mu}_{b,h}\|^2_{H^{-2}(\Omega)} \\
& + \|\hat{y} - \bar{y}_h\|^2_{L^2(\Omega)}\sigma^2 + \|y^h - \bar{y}_h\|^2_{L^2(\Omega)} \\
& + (\alpha\bar{u}_h + \bar{p}_h, \hat{u} - \bar{u}_h)\sigma + \|\eta_{\mathrm{vi}}\|^2_{L^2(\Omega)} \\
& + \langle\bar{\mu}_{a,h} - \bar{\mu}_{b,h},\ \hat{y} - \bar{y}_h\rangle\sigma + \|\bar{\mu}_{a,h} - \bar{\mu}_{b,h}\|_{M(\Omega)}\|y^h - \bar{y}_h\|_{L^\infty(\Omega)}\Big\}.
\end{aligned}
$$
(3.16)

The value of σ can be bounded according to Lemma 3.1 by

$$\sigma \leq \tau^{-1}(\|y^h - \bar{y}_h\|_{L^\infty(\Omega)} + e_{\mathrm{sc}}),$$

where e_{sc} is the state constraint violation defined in (3.2). Using this estimate in (3.16), the claim follows.   □

In Sect. 4, we will prove convergence to zero of the upper bound (3.15) if the discrete quantities converge in some sense to solutions of the optimality system in Theorem 2.3.

3.5 Localized a-posteriori error estimates

In the previous sections, we developed error bounds for the discretization error. These bounds contain terms that are still not fully accessible. In particular, it needs to be specified, how the L^2- and L^∞-discretization errors of the states as well as the residual of the adjoint equation can be calculated.

For the L^2-error of the states, we have the following result, which is a standard estimate, see e.g. [5]. Recall that $\bar{y}_h$ is the solution of the discretized equation (2.6) with right-hand side $\bar{u}_h$, while y^h is the solution of the elliptic equation (3.1) with the same right-hand side $\bar{u}_h$.

Lemma 3.15 *There is a constant $c > 0$ depending on Ω, the polynomial degree l, and the shape regularity of the triangulation such that*

$$\|y^h - \bar{y}_h\|_{L^2(\Omega)} \le c\, \eta^2_{\text{state},L^2}$$

with $\eta^2_{\text{state},L^2} = \sum_{T \in \mathcal{T}_h} \eta^2_{T,\text{state},L^2}$ and

$$\eta^2_{T,\text{state},L^2} = \left(h_T^4 \|\Delta \bar{y}_h - \bar{y}_h + \bar{u}_h\|^2_{L^2(T)} + h_T^3 \left\|\left[\frac{\partial \bar{y}_h}{\partial n}\right]\right\|^2_{L^2(\partial T \setminus \Gamma)} \right).$$

Here, $\left[\frac{\partial \bar{y}_h}{\partial n}\right]$ denotes the jump of the normal derivative across interior edges.

To estimate the L^∞-error we use the following reliable and efficient error estimator from [21].

Lemma 3.16 *There is a constant $c > 0$ depending on Ω, the polynomial degree l, and the shape regularity of the triangulation such that*

$$\|y^h - \bar{y}_h\|_{L^\infty(\Omega)} \le c\, \eta_{\text{state},L^\infty}$$

with $\eta_{\text{state},L^\infty} = \max_{T \in \mathcal{T}_h} \eta_{T,\text{state},L^\infty}$ and

$$\eta_{T,\text{state},L^\infty} = |\log h_{\min}|^2 \left(h_T^2 \|\Delta \bar{y}_h - \bar{y}_h + \bar{u}_h\|_{L^\infty(T)} + h_T \left\|\left[\frac{\partial \bar{y}_h}{\partial n}\right]\right\|_{L^\infty(\partial T \setminus \Gamma)} \right),$$

where $h_{\min} := \min_{T \in \mathcal{T}_h} h_T$.

It remains to describe the estimation of the H^{-2}-residual of the adjoint equation.

Lemma 3.17 *There is a constant $c > 0$ depending on Ω, the polynomial degree l, and the shape regularity of the triangulation such that*

$$\|-A^*\bar{p}_h + \bar{y}_h - y_d - \bar{\mu}_{a,h} + \bar{\mu}_{b,h}\|_{H^{-2}(\Omega)} \le c\,\eta^2_{\text{adjoint}, L^2}$$

with $\eta^2_{\text{adjoint}, L^2} = \sum_{T \in \mathcal{T}_h} \eta^2_{T, \text{adjoint}, L^2}$ and

$$\eta^2_{T, \text{adjoint}, L^2} = \left(h_T^4 \|\Delta\bar{p}_h - \bar{p}_h + \bar{y}_h - y_d\|^2_{L^2(T)} + h_T^3 \left\| \left[\frac{\partial\bar{p}_h}{\partial n} \right] \right\|^2_{L^2(\partial T \backslash \Gamma)} \right).$$

Proof For a test function $\phi \in V = H_0^1(\Omega) \cap H^2(\Omega)$ we will employ the Lagrange interpolation $I_h\phi$, which has the property $\phi(x_i) = (I_h\phi)(x_i)$ for all nodes x_i. Due to the assumptions on V_h we get $I_h\phi \in V_h$. Since $\bar{p}_h$ solves the discrete adjoint equation (2.9), we have

$$
\begin{aligned}
\|-A^*\bar{p}_h &+ \bar{y}_h - y_d - \bar{\mu}_{a,h} + \bar{\mu}_{b,h}\|_{H^{-2}(\Omega)} \\
&= \sup_{\phi \in V,\, \|\phi\|_{H^2(\Omega)} = 1} \langle -A^*\bar{p}_h + \bar{y}_h - y_d - \bar{\mu}_{a,h} + \bar{\mu}_{b,h}, \phi \rangle \\
&= \sup_{\phi \in V,\, \|\phi\|_{H^2(\Omega)} = 1} \langle -A^*\bar{p}_h + \bar{y}_h - y_d - \bar{\mu}_{a,h} + \bar{\mu}_{b,h}, \phi - I_h\phi \rangle.
\end{aligned}
$$

The multipliers $\bar{\mu}_{a,h}$ and $\bar{\mu}_{b,h}$ are linear combinations of Dirac measures concentrated in the mesh nodes, which implies

$$\langle -\bar{\mu}_{a,h} + \bar{\mu}_{b,h}, \phi - I_h\phi \rangle = 0.$$

Hence, we obtain

$$
\begin{aligned}
\|-A^*\bar{p}_h &+ \bar{y}_h - y_d - \bar{\mu}_{a,h} + \bar{\mu}_{b,h}\|_{H^{-2}(\Omega)} \\
&= \sup_{\phi \in V,\, \|\phi\|_{H^2(\Omega)} = 1} \langle -A^*\bar{p}_h + \bar{y}_h - y_d, \phi - I_h\phi \rangle.
\end{aligned}
$$

Following the arguments in [2] we find the local representation for the residual.   □

Although this result of [2] is formulated only for $n = 2, l = 1$, and a single Dirac measure, the proofs carry over one-to-one to the case considered here: $n \in \{2, 3\}$, general FE-space V_h with $l \ge 1$, right-hand side consists of linear combination of Dirac measures concentrated in the nodes.

In these lemmata, we cited only the reliability estimates (i.e. upper error bounds). For all three estimators also lower error bounds (efficiency estimates) are available.

The localization of the estimator of the error in the variational inequality is an obvious choice. Let us define

$$\eta^2_{T, \text{control}, L^2} := \eta_{\text{vi}}|_T,$$

where $\eta_{\text{vi}} = \eta(\bar{u}_h, \bar{p}_h)$ is the function constructed in Lemma 3.7.

Combining the estimates of this section with the result of Theorem 3.14, we get our main result, which is the localized a-posteriori error estimate.

Theorem 3.18 *Let $(\bar{y}, \bar{u})$ be the solution of the continuous optimal control problem (1.1)–(1.4). Let $(\bar{y}_h, \bar{u}_h, \bar{p}_h, \bar{\mu}_{a,h}, \bar{\mu}_{b,h})$ satisfy (2.7)–(2.8), (2.10).*
Then there is a constant $c > 0$ depending on Ω, τ, and the shape regularity of the triangulation, and a weight $\omega_{\infty,h} > 0$ such that

$$\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)}^2 + \|\bar{y}_h - \bar{y}\|_{L^\infty(\Omega)}^2 \le c\,\alpha^{-1}\left(\eta_{L^2}^2 + \omega_{\infty,h}\,\eta_{\text{state},L^\infty} + e_{\text{state}}\right)$$

with $\eta_{L^2}^2 = \sum_{T \in \mathcal{T}_h} \eta_{T,L^2}^2$ and

$$\eta_{T,L^2}^2 = \eta_{T,\text{state},L^2}^2 + \eta_{T,\text{control},L^2}^2 + \alpha^{-1}\eta_{T,\text{adjoint},L^2}^2.$$

The weight $\omega_{\infty,h}$ depends on the discrete quantities $(\bar{y}_h, \bar{u}_h, \bar{p}_h, \bar{\mu}_{a,h}, \bar{\mu}_{b,h})$. The mapping from $(\bar{y}_h, \bar{u}_h, \bar{p}_h, \bar{\mu}_{a,h}, \bar{\mu}_{b,h})$ to $\omega_{\infty,h}$ is bounded from $C(\bar{\Omega}) \times L^2(\Omega) \times L^2(\Omega) \times M(\Omega) \times M(\Omega)$ to $\mathbb{R}$.

Proof The result follows directly from Theorem 3.14 and Lemmata 3.15, 3.16, and 3.17. The quantity $\omega_{\infty,h}$ is given by

$$\omega_{\infty,h} = \max\left((\alpha\bar{u}_h + \bar{p}_h, \hat{u} - \bar{u}_h) + \langle\bar{\mu}_{a,h} - \bar{\mu}_{b,h}, \hat{y} - \bar{y}_h\rangle,\ 0\right) + \|\bar{\mu}_{a,h} - \bar{\mu}_{b,h}\|_{M(\Omega)}$$
$$+ \left(\|\hat{u} - \bar{u}_h\|_{L^2(\Omega)}^2 + \|\hat{y} - \bar{y}_h\|_{L^2(\Omega)}^2\right)\|y^h - \bar{y}_h\|_{L^\infty(\Omega)},$$

cf. the definition of $r_{\text{state},L^\infty}^{(1)}$ and $r_{\text{state},L^\infty}^{(2)}$ in Theorem 3.14. The mapping $\bar{u}_h \mapsto y^h$ is bounded from $L^2(\Omega)$ to $L^\infty(\Omega)$ by Lemma 2.1, which proves the claimed boundedness of $\omega_{\infty,h}$.   □

3.6 Marking strategy

It remains to describe, how to mark elements for refinement. Here, we follow the common strategy to mark elements that have relatively large local error indicators. In our case, see Theorem 3.18, the error indicator contains two terms with different accumulation properties: $\eta_{L^2}^2 = \sum_{T \in \mathcal{T}_h} \eta_{T,L^2}^2$ and $\eta_{\text{state},L^\infty} = \max_{T \in \mathcal{T}_h}(\eta_{T,\text{state},L^\infty} + e_{T,\text{state}})$, where $e_{T,\text{state}}$ is the restriction of e_{state} to T.

As marking strategy we employ the one used in [21]. Let us define

$$\eta_2^2 := \eta_{L^2}^2$$
$$\eta_\infty := \omega_{\infty,h}\,\eta_{\text{state},L^\infty} + e_{\text{state}}$$
$$\eta := \max(\eta_2,\ \eta_\infty).$$

We choose an error indicator η_i if it is relatively large compared to the total error, that is if $\eta_i \ge \theta_1\eta$, $i \in \{2, \infty\}$. For a chosen error indicator η_i, we mark elements by the

maximum strategy, that is, elements $\hat{T}$ with $\eta_{\hat{T},i} \geq \theta_2 \max_{T \in \mathcal{T}_h} \eta_{T,i}$ are marked for refinement. Here, the parameters θ_1, θ_2 are taken from $(0, 1)$. In our computations we used $\theta_1 = 0.2, \theta_2 = 0.8$.

4 Convergence of error bound

In this section we will prove the convergence to zero of the error bound of Theorem 3.14 if the solution of the discrete system converges in the following sense.

Assumption 4 Let a sequence of meshes $\mathcal{T}_{h_k}$ with associated solutions of the discrete problem $\{(\bar{y}_{h_k}, \bar{u}_{h_k}, \bar{p}_{h_k}, \bar{\mu}_{a,h_k}, \bar{\mu}_{b,h_k})\}$ be given. Let us assume that we have the following properties of this sequence:

(i)   The sequence $(\bar{y}_{h_k}, \bar{u}_{h_k})$ converges strongly to $(\bar{y}, \bar{u})$ in $C(\bar{\Omega}) \times L^2(\Omega)$.
(ii) The sequence $(\bar{p}_{h_k}, \bar{\mu}_{a,h_k}, \bar{\mu}_{b,h_k})$ is bounded in $W^{1,q}(\Omega) \times M(\Omega) \times M(\Omega)$ with $\frac{2n}{2+n} < q < \frac{n}{n-1}$.
(iii) For each subsequence $(\bar{p}_{h_{k'}}, \bar{\mu}_{a,h_{k'}}, \bar{\mu}_{b,h_{k'}})$ with $\bar{p}_{h_{k'}} \rightharpoonup \bar{p}$ in $W^{1,q}(\Omega)$, $q > \frac{2n}{2+n}$, and $(\bar{\mu}_{a,h_{k'}}, \bar{\mu}_{b,h_{k'}}) \overset{*}{\rightharpoonup} (\bar{\mu}_a, \bar{\mu}_b)$ in $M(\Omega) \times M(\Omega)$ the limit element $(\bar{p}, \bar{\mu}_a, \bar{\mu}_b)$ is a Lagrange multiplier to $(\bar{y}, \bar{u})$, i.e. the system (2.2)–(2.4) is satisfied.

Please note, that the requirements of this assumption are satisfied for a uniform refinement of the mesh. Boundedness of multipliers $\mu_{a,h}, \mu_{b,h}$ in $M(\Omega)$ can be proven following see e.g. [15, Chapter 3]. The uniform boundedness of discrete adjoints p_h then follows from [5, Theorem 8.5.3]. Let us emphasize that convergence of the mesh size $h \to 0$ is not explicitly required in Assumption 4.

Let us recall the error representation of Theorem 3.14, which reads

$$\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)}^2 + \|\bar{y}_h - \bar{y}\|_{L^\infty(\Omega)}^2 \leq c\Big(r_{\text{state},L^\infty}^{(1)} + r_{\text{state},L^\infty}^{(2)} + r_{\text{state},L^2}^2 + r_{\text{adjoint},L^2}^2 + r_{\text{control},L^2}^2 + e_{\text{state}}\Big).$$

Lemma 4.1 *Let Assumption 4 be satisfied. Then it holds*

$$r_{\text{state},L^\infty}^{(1)} + r_{\text{state},L^\infty}^{(2)} + r_{\text{state},L^2}^2 + r_{\text{adjoint},L^2}^2 + e_{\text{state}} \to 0$$

for $k \to \infty$.

Proof Let us define y^{h_k} to be the solution of (3.1) to the control $\bar{u}_{h_k}$. Then we have

$$\|y^{h_k} - \bar{y}_{h_k}\|_{L^\infty(\Omega)} \leq \|y^{h_k} - \bar{y}\|_{L^\infty(\Omega)} + \|\bar{y} - \bar{y}_{h_k}\|_{L^\infty(\Omega)}$$
$$\leq C\|\bar{u}_{h_k} - \bar{u}\|_{L^2(\Omega)} + \|\bar{y} - \bar{y}_{h_k}\|_{L^\infty(\Omega)}.$$

which proves that $r_{\text{state},L^\infty}^{(1)}$, $r_{\text{state},L^\infty}^{(2)}$, and similarly r_{state,L^2} converge to zero under the Assumption 4.

Because of $\bar{y}_{h_k} \to \bar{y}$ in $C(\bar{\Omega})$ and the feasibility of $\bar{y}$ with respect to the control constraints, the state constraint violation e_{sc} and consequently e_{state} tend to zero.

Let weak (weak*) converging subsequences $(\bar{p}_{h_{k'}}, \bar{\mu}_{a,h_{k'}}, \bar{\mu}_{b,h_{k'}})$ be given. Then by compact embeddings and after extracting another subsequence $\bar{p}_{h_{k''}} \to \bar{p}$ in $L^2(\Omega)$ and $(\bar{\mu}_{a,h_{k''}}, \bar{\mu}_{b,h_{k''}}) \to (\bar{\mu}_a, \bar{\mu}_b)$ in $H^{-2}(\Omega)$. Hence $r_{adjoint,L^2} \to 0$ for this subsequence. Since the subsequence (k') was chosen arbitrarily, it follows $r_{adjoint,L^2} \to 0$ for $k \to \infty$.   □

The discussion of the estimator $r_{control,L^2}$ is more involved, and thus we state and prove it separately. For convenience, let us recall its definition. In Theorem 3.14 we set $r_{control,L^2} := \|\eta_{vi}\|_{L^2(\Omega)}$, where η_{vi} was given by Lemma 3.7 and defined by

$$\eta_{vi}^2 = \frac{1}{\alpha}\chi_{\Omega_{0,h}}(\alpha\bar{u}_h + \bar{p}_h)^2 + \chi_{\Omega_{a,h}}(\alpha\bar{u}_h + \bar{p}_h)(\bar{u}_h - u_a)$$
$$+ \chi_{\Omega_{b,h}}(\alpha\bar{u}_h + \bar{p}_h)(\bar{u}_h - u_b),$$

where the sets $\Omega_{0,h}, \Omega_{a,h}, \Omega_{b,h}$ were defined as

$$\Omega_{0,h} = \left\{ x \in \Omega : \ \bar{u}_h(x), -\frac{1}{\alpha}\bar{p}_h(x) \in (u_a, u_b) \right.$$
$$\text{or } \bar{u}_h(x) = u_a, \alpha\bar{u}_h(x) + \bar{p}_h(x) < 0$$
$$\left. \text{or } \bar{u}_h(x) = u_b, \alpha\bar{u}_h(x) + \bar{p}_h(x) > 0 \right\}$$

and

$$\Omega_{a,h} = \left\{ x \in \Omega : \bar{u}_h(x) \in (u_a, u_b), -\frac{1}{\alpha}\bar{p}_h(x) \leq u_a \right\}$$
$$\Omega_{b,h} = \left\{ x \in \Omega : \bar{u}_h(x) \in (u_a, u_b), -\frac{1}{\alpha}\bar{p}_h(x) \geq u_b \right\}.$$

Lemma 4.2 *Let Assumption 4 be satisfied. Then it holds*

$$r_{control,L^2} \to 0.$$

Proof Since the controls $\bar{u}_{h_k}$ are feasible for the discrete problem, they are uniformly bounded in $L^\infty(\Omega)$. Hence, the sequence $\bar{u}_{h_k}$ converges to $\bar{u}$ in $L^s(\Omega)$ for all $2 \leq s < \infty$. Moreover, as argued above, each weakly converging subsequence $\bar{p}_{h_{k'}}$ has a strongly converging subsequence $\bar{p}_{h_{k''}} \to \bar{p}$ in $L^2(\Omega)$. By Assumption 4, we have that the variational inequality (2.4) is satisfied.

In the course of the proof, we will bound $\|\eta_{vi}\|_{L^2(\Omega)}$ in terms of $\|\bar{p}_{h_{k''}} - \bar{p}\|_{L^2(\Omega)}$ and $\|\bar{u}_{h_{k''}} - \bar{u}\|_{L^s(\Omega)}, s > 2$. To simplify the notation, we will drop the index k''.

At first, let us note that

$$\eta_{vi}(x)^2 \leq \frac{1}{\alpha}|\alpha\bar{u}_h(x) + \bar{p}_h(x)|^2 \tag{4.1}$$

follows directly from the definition of $\eta_{vi}, \Omega_{a,h}$, and $\Omega_{b,h}$, confer also (3.7).

Now, we will discuss upper bounds of η_{vi} on different subsets of Ω.

Case (1): $\Omega_1 = \{x \in \Omega : \alpha \bar{u}(x) + \bar{p}(x) = 0\}$.

Using (4.1) it follows directly

$$\|\eta_{vi}\|^2_{L^2(\Omega_1)} \le \alpha^{-1} \|\alpha \bar{u}_h + \bar{p}_h - (\alpha \bar{u} + \bar{p})\|^2_{L^2(\Omega_1)}$$
$$\le \alpha^{-1} (\alpha \|\bar{u}_h - \bar{u}\|_{L^2(\Omega)} + \|\bar{p}_h - \bar{p}\|_{L^2(\Omega)})^2. \qquad (4.2)$$

Case (2_a): $\Omega_{2,a} = \{x \in \Omega : \bar{u}_h(x) = \bar{u}(x) = u_a\}$. Here, we have

$$\|\eta_{vi}\|^2_{L^2(\Omega_{2,a})} = \alpha^{-1} \|(\alpha \bar{u}_h + \bar{p}_h)^-\|^2_{L^2(\Omega_{2,a})}$$
$$= \alpha^{-1} \left\|(\alpha \bar{u}_h + \bar{p}_h)^- - (\alpha \bar{u} + \bar{p})^-\right\|^2_{L^2(\Omega_{2,a})}$$
$$\le \alpha^{-1} (\alpha \|\bar{u}_h - \bar{u}\|_{L^2(\Omega)} + \|\bar{p}_h - \bar{p}\|_{L^2(\Omega)})^2.$$

Case (2_b): $\Omega_{2,b} = \{x \in \Omega : \bar{u}_h(x) = \bar{u}(x) = u_b\}$. Analogous to *Case ($2_a$)*.

Case (3_a): $\Omega_{3,a} = \{x \in \Omega : \bar{u}(x) = u_a\} \cap \Omega_{a,h}$. Here, we have by definition of η_{vi}

$$\eta_{vi}(x)^2 = (\alpha \bar{u}_h + \bar{p}_h)(\bar{u}_h - u_a) = (\alpha \bar{u}_h + \bar{p}_h)(\bar{u}_h - \bar{u}).$$

Hence, it holds $\|\eta_{vi}\|^2_{L^2(\Omega_{3,a})} \le \|\alpha \bar{u}_h + \bar{p}_h\|_{L^2(\Omega)} \|\bar{u}_h - \bar{u}\|_{L^2(\Omega)}$.

Case (3_b): $\Omega_{3,b} = \{x \in \Omega : \bar{u}(x) = u_b\} \cap \Omega_{b,h}$. Analogous to *Case ($3_a$)*.

Case (4_a): $\Omega_{4,a} = \{x \in \Omega : \bar{u}_h(x), -\frac{1}{\alpha} \bar{p}_h(x) \in (u_a, u_b), \bar{u}(x) = u_a, -\frac{1}{\alpha} \bar{p}(x) < u_a\}$. The inequality $-\frac{1}{\alpha} \bar{p}(x) < u_a = \bar{u}(x) < -\frac{1}{\alpha} \bar{p}_h(x)$ implies

$$|\alpha \bar{u}_h + \bar{p}_h| \le \alpha |\bar{u}_h - \bar{u}| + \alpha |\bar{u} - (-\alpha^{-1} \bar{p}_h)| \le \alpha |\bar{u}_h - \bar{u}| + |-\bar{p} - (-\bar{p}_h)|$$

which proves with (4.1)

$$\|\eta_{vi}\|^2_{L^2(\Omega_{4,a})} \le \frac{1}{\alpha} \|\alpha \bar{u}(x) + \bar{p}(x)\|^2_{L^2(\Omega_{4,a})} \le \|\bar{p}_h - \bar{p}\|^2_{L^2(\Omega_{4,a})}.$$

Case (4_b): $\Omega_{4,b} = \{x \in \Omega : \bar{u}_h(x), -\frac{1}{\alpha} \bar{p}_h(x) \in (u_a, u_b), \bar{u}(x) = u_b, -\frac{1}{\alpha} \bar{p}(x) < u_b\}$. Analogous to *Case ($4_a$)*.

Case (5_a): $\Omega_{5,a} = \{x \in \Omega : \bar{u}(x) = u_a\} \cap \Omega_{b,h}$. Here, we have $-\frac{1}{\alpha} \bar{p}(x) \le u_a < u_b < -\frac{1}{\alpha} \bar{p}_h(x)$. By Chebyshev's inequality, we obtain

$$|\Omega_{5,a}| \le \left|\left\{x \in \Omega : \alpha^{-1} |\bar{p}_h(x) - \bar{p}(x)| \ge |u_b - u_a|\right\}\right|$$
$$\le \int_\Omega \frac{|\bar{p}_h(x) - \bar{p}(x)|^2}{\alpha^2 |u_b - u_a|^2}$$
$$\le \frac{1}{(\alpha |u_b - u_a|)^2} \|\bar{p}_h - \bar{p}\|^2_{L^2(\Omega)}.$$

Together with the definition η_{vi} on $\Omega_{b,h}$, we find

$$\|\eta_{vi}\|^2_{L^2(\Omega_{5,a})} \leq |\Omega_{5,a}|^{1/2}\|\alpha\bar{u}_h + \bar{p}_h\|_{L^2(\Omega)}\|\bar{u}_h - u_b\|_{L^\infty(\Omega)} \leq c\|\bar{p}_h - \bar{p}\|_{L^2(\Omega)}.$$

Case (5$_b$): $\Omega_{5,b} = \{x \in \Omega : \bar{u}(x) = u_b\} \cap \Omega_{a,h}$. Analogous to *Case (5$_a$)*.

Case (6): $\Omega_6 = \{x \in \Omega : \bar{u}(x), \bar{u}_h(x) \in \{u_a, u_b\}, \bar{u}(x) \neq \bar{u}_h(x)\}$. That is, here the control constraints are active at $\bar{u}$ and $\bar{u}_h$ but both are not equal. Similarly as in *Case (5$_a$)* we estimate

$$\begin{aligned}
|\Omega_6| &\leq \left|\{x \in \Omega : |\bar{u}(x) - \bar{u}_h(x)| \geq |u_b - u_a|\}\right| \\
&\leq \int_\Omega \frac{|\bar{u}_h(x) - \bar{u}(x)|^s}{|u_b - u_a|^s} \\
&\leq \frac{1}{|u_b - u_a|^s}\|\bar{u}_h - \bar{u}\|^s_{L^s(\Omega)},
\end{aligned}$$

which yields due to $\Omega_6 \subset \Omega_{0,h}$

$$\|\eta_{vi}\|^2_{L^2(\Omega_6)} \leq \frac{1}{\alpha}|\Omega_6|^{1-2/s}\|\alpha\bar{u}_h + \bar{p}_h\|^2_{L^s(\Omega)} \leq C\|\bar{u}_h - \bar{u}\|^{s-2}_{L^s(\Omega)}.$$

Let us argue that the splitting introduced by the cases above covers Ω. Due to first order optimality conditions, Ω can be divided in sets, where $\alpha\bar{u} + \bar{p} = 0$ and $\bar{u} \in \{u_a, u_b\}$. The first possibility is covered by *Case (1)*. The case that both $\bar{u}$ and $\bar{u}_h$ are at the bounds is contained in *Case (2)* and *Case (6)*. Now it remains to cover the set, where $\bar{u} \in \{u_a, u_b\}$ and $\bar{u}_h \in (u_a, u_b)$. The subset, where $-\frac{1}{\alpha}\bar{p}_h$ is not in (u_a, u_b), is discussed in *Case (3)* and *Case (5)*. And *Case (4)* covers the subset, where $-\frac{1}{\alpha}\bar{p}_h$ is in (u_a, u_b).

Summing up all the estimates, we find the convergence $\eta_{vi} \to 0$ in $L^2(\Omega)$ for the subsequence (k'') chosen above. This implies that for every subsequence of $(\bar{u}_{h_k}, \bar{p}_{h_k})$ we can choose a subsequence such that the corresponding quantity η_{vi} converges to zero, which finishes the proof. □

As a consequence of these results we obtain the main result of this section.

Theorem 4.3 *Let Assumption 4 be satisfied. Then the error bound given by Theorem 3.14 converges to zero for $k \to \infty$, e.g.*

$$\left(r^{(1)}_{state,L^\infty} + r^{(2)}_{state,L^\infty} + r^2_{state,L^2} + r^2_{adjoint,L^2} + r^2_{control,L^2} + e_{state}\right) \to 0.$$

Let us now comment on different available error estimators for the error in the variational inequality connected to the control constraints. To the best of our knowledge, a convergence result similar to Lemma 4.2 is not available for error estimators in the literature.

Let us first compare our findings to the error estimator as considered by Krumbiegel and Rösch [17]. Their error estimator η_{KR} coincides with η_{vi} as developed here except for the definition on the sets $\Omega_{a,h}$, $\Omega_{b,h}$, i.e.

$$\eta_{\mathrm{KR}} = \chi_{\Omega_{0,h}} \eta_{\mathrm{vi}} + \frac{1}{\alpha} \chi_{\Omega_{a,h} \cup \Omega_{b,h}} (\alpha \bar{u}_h + \bar{p}_h)^2.$$

which implies $\|\eta_{\mathrm{vi}}\|_{L^2(\Omega)}^2 \leq \|\eta_{\mathrm{KR}}\|_{L^2(\Omega)}^2$ by (3.7). The error estimator η_{KR} does not allow a convergence proof comparable to Lemma 4.2. To this end consider the following example: let the constant functions $u_a = 0$, $u_b = 1$, $\alpha = 1$, $\bar{u} = 0$, $\bar{p} = \bar{p}_h = 1$, and $\bar{u}_h = h$ be given. That means, the values are chosen in such a way that $\Omega_{a,h} = \Omega$ and correspond to the *Case (3$_a$)* in the proof of Lemma 4.2. Our error estimator gives $\|\eta_{\mathrm{vi}}\|_{L^2(\Omega)}^2 = (\alpha \bar{u}_h + \bar{p}_h, \bar{u}_h - u_a) = (\alpha h + 1)h |\Omega| \to 0$ as $h \to 0$. The estimator of [17] yields $\|\eta_{\mathrm{KR}}\|_{L^2(\Omega)}^2 = \frac{1}{\alpha} \|\alpha h + 1\|_{L^2(\Omega)}^2$, which does not converge to zero as $h \to 0$. Similar situations as in the example will occur if one uses interior point methods to get rid of the control constraints in the discrete optimization problem.

Second, let us comment on the error estimator analyzed by Hintermüller et al. [13]. There an efficient and reliable error estimator for optimal control problems with a lower control bound is developed. They prove that the estimator is equivalent (up to higher order terms) to the error

$$\|\bar{y} - \bar{y}_h\|_{H^1(\Omega)} + \|\bar{p} - \bar{p}_h\|_{H^1(\Omega)} + \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} + \|\sigma - \sigma_h\|_{L^2(\Omega)},$$

with $\sigma := \alpha \bar{u} + \bar{p}$ and $\sigma_h := \alpha \bar{u}_h + \Pi_h \bar{p}_h$. However, it is not clear, whether it holds $\|\sigma - \sigma_h\|_{L^2(\Omega)} \to 0$ if $(\bar{u}_h, \bar{p}_h)$ converges to $(\bar{u}, \bar{p})$. Indeed, for *Case (3$_a$)* in the proof of Lemma 4.2 one has the following. If $(\bar{u}_h, \bar{p}_h)$ are solutions of the discrete optimal control problem it holds $\alpha \bar{u}_h + \Pi_h \bar{p}_h = 0$ on this set, since $\bar{u}_h$ is not at the control bounds. This implies that

$$\|\sigma - \sigma_h\|_{L^2(\Omega)} = \|(\alpha \bar{u} + \bar{p}) - (\alpha \bar{u}_h + \Pi_h \bar{p}_h)\|_{L^2(\Omega_{3,a})} = \|\alpha \bar{u} + \bar{p}\|_{L^2(\Omega_{3,a})}.$$

However, since $\bar{u} = u_a$ holds on $\Omega_{3,a}$ by definition, this quantity is in general nonzero. In order to prove $\|\sigma - \sigma_h\|_{L^2(\Omega)} \to 0$, one has to prove in addition that the measure of $\Omega_{3,a}$ tends to zero.

Finally, we comment on the error estimator for control constrained optimal control problems as considered by Li et al. [18]. The error estimator developed there converges to zero under the assumption that $(\bar{y}_h, \bar{u}_h, \bar{p}_h)$ converges to $(\bar{y}, \bar{u}, \bar{p})$, an assumption on the regularity of the active sets, and the assumption $h \to 0$. Clearly our convergence result Lemma 4.2 holds under weaker assumptions on $\bar{u}_h, \bar{p}_h$.

5 Numerical experiment

Let us report on numerical results with adaptive refinement using the error estimator developed in the present article.

5.1 Example 1

The data of this example is taken from [7]. It was originally posed for $\Omega = B_1(0)$. We modified it to work with $\Omega = (-1, 2)^2$.

The data of the problem is given in polar coordinates. For convenience we define $r = \sqrt{x_1^2 + x_2^2}$. Let us set

$$\bar{u}(r) = -\frac{1}{2\pi\alpha}\chi_{\{r\leq 1\}}(\log r + r^2 - r^3),$$

$$\bar{y}(r) = \frac{1}{2\pi\alpha}\chi_{\{r\leq 1\}}\left(\frac{r^2}{4}(\log r - 2) + \frac{r^3}{4} + \frac{1}{4}\right),$$

$$\bar{p}(r) = -\alpha\bar{u}(r),$$

$$\bar{\mu}_a(r) = \delta_0(r),$$

$$f(r) = \frac{1}{8\pi}\chi_{\{r\leq 1\}}(4 - 9r + 4r^2 - 4r^3),$$

$$y_d(r) = \bar{y}(r) + \frac{1}{2\pi}\chi_{\{r\leq 1\}}(4 - 9r),$$

$$y_a(r) = \frac{1}{2\pi\alpha}\left(\frac{1}{4} - \frac{r}{2}\right),$$

$$\alpha = 1.$$

The problem features one lower state constraint, there are no upper state constraint and no control constraints given.

One can verify that $(\bar{y}, \bar{u}, \bar{p}, \bar{\mu}_a)$ is the solution of the problem: Minimize $J(y, u)$ given by

$$J(y, u) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}$$

subject to

$$-\Delta y = u + f \text{ in } \Omega$$
$$y = 0 \text{ on } \Gamma,$$

and

$$y_a \leq y \text{ in } \Omega.$$

Moreover, one can verify that

$$\hat{u}(r) = \chi_{\{r\leq 1\}}(-12 + 18r) \qquad \hat{y}(r) = \chi_{\{r\leq 1\}}(1 + 3r^2 - 2r^3)$$

fulfills Assumption 2 with $\tau = 0.24999$.

Due to the special structure of the problem, the dual quantities are uniquely determined. Please note, that the adjoint state has a pronounced singularity. Since the data of the problem are smooth, there is no chance to perform an a-priori mesh refinement to resolve the singularity.
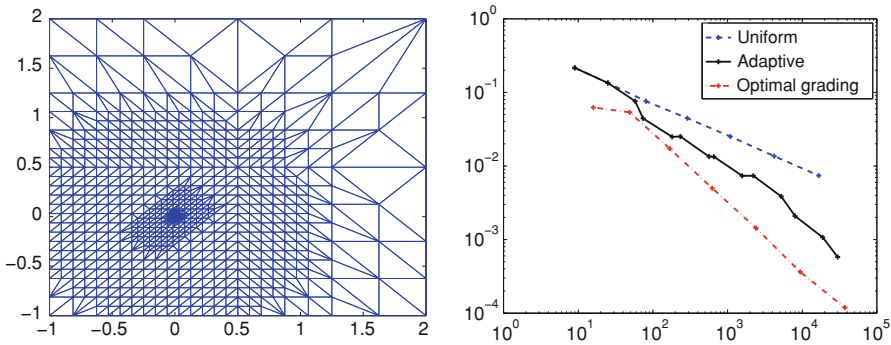
Fig. 1 Example 1: adaptively refined mesh; comparison of $\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)}$ versus number of unknowns for uniform, adaptive and uniform with grading refinement

We discretize states, adjoints, and controls by continuous and piecewise linear elements, which corresponds to the choice $l = 1$.

We computed discrete solutions for different kinds of mesh refinement. For uniform and adaptive refinement, we ensured that the point $x = (0, 0)$ cannot be a node of the grid for any refined mesh to avoid superconvergence effects. This is achieved with a coarse grid mesh generated as a uniform triangulation with 8 triangles. The results of these computations are then compared to results computed with an optimally graded mesh according to [1]. There, in each uniform refinement step, the newly added grid points are graded to the point $(0, 0)$ with parameter $\mu = 0.5$. Moreover, the coarse grid was built from 18 triangles such that the point $(0, 0)$ is a node of that grid, and consequently is a node of all refined grids. This is an a-priori refinement, which is only possible, since the solution of this problem is already known.

Figure 1 shows the adaptively refined mesh after five refinement steps. It shows local refinement around the singularity at $(0, 0)$, which appears only in the adjoint equation and thus is not identifiable a-priori. The right-hand plot compares the error $\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)}$ for the different kind of discretizations. Here, the adaptive mesh refinement leads to a better approximation of the solution with respect to the number of unknowns than uniform refinement. However, adaptive refinement is still worse than the optimal a-priori refinement. Of course, the optimal a-priori mesh-grading is only available because we know the solution of the continuous problem. Nevertheless, the optimal mesh-grading can serve as benchmark for adaptive refinement, which ideally should yield the same performance.

Let us now discuss the efficiency of our error estimators. To this end, the efficiency index

$$I_{\text{eff}} = \frac{\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}}{\eta}$$

is depicted in Tables 1, 2, and 3. Ideally, the efficiency index stays constant during the mesh refinement process. We cannot expect it to be equal to one, which would be the ideal case, due to the unknown constants in the localized error estimates in lemmata 3.15, 3.16, and 3.17. As one can see from Table 1, the efficiency index is

Table 1 Example 1: results for uniform refinement

N	$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}$	η	I_{eff}
9	2.1598×10^{-1}	8.8802	2.4321×10^{-2}
25	1.3457×10^{-1}	2.0605	6.5312×10^{-2}
81	7.5782×10^{-2}	9.5844×10^{-1}	7.9069×10^{-2}
289	4.4661×10^{-2}	5.7608×10^{-1}	7.7526×10^{-2}
1089	2.5174×10^{-2}	3.3665×10^{-1}	7.4777×10^{-2}
4225	1.3598×10^{-2}	1.9939×10^{-1}	6.8197×10^{-2}
16641	7.4047×10^{-3}	1.1143×10^{-1}	6.6449×10^{-2}

Table 2 Example 1: results for adaptive refinement

N	$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}$	η	I_{eff}
9	2.1598×10^{-1}	2.6039	8.2944×10^{-2}
25	1.3457×10^{-1}	4.8494×10^{-1}	2.7750×10^{-1}
58	7.5789×10^{-2}	3.0258×10^{-1}	2.5048×10^{-1}
74	4.4358×10^{-2}	6.1424×10^{-1}	7.2216×10^{-2}
181	2.4899×10^{-2}	5.2401×10^{-1}	4.7516×10^{-2}
234	2.5190×10^{-2}	2.9914×10^{-1}	8.4207×10^{-2}
564	1.3438×10^{-2}	3.1367×10^{-1}	4.2840×10^{-2}
657	1.3375×10^{-2}	1.8474×10^{-1}	7.2400×10^{-2}
1573	7.3654×10^{-3}	1.5810×10^{-1}	4.6588×10^{-2}
2217	7.3698×10^{-3}	9.7562×10^{-2}	7.5540×10^{-2}
5246	3.8560×10^{-3}	8.0546×10^{-2}	4.7874×10^{-2}
8020	2.0885×10^{-3}	6.3861×10^{-2}	3.2703×10^{-2}
18839	1.0715×10^{-3}	5.0336×10^{-2}	2.1286×10^{-2}
29853	5.8029×10^{-4}	3.8987×10^{-2}	1.4884×10^{-2}

Table 3 Example 1: results for uniform refinement with optimal mesh-grading

N	$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}$	η	I_{eff}
16	6.2462×10^{-2}	1.0628	5.8771×10^{-2}
49	5.3629×10^{-2}	9.7516×10^{-1}	5.4995×10^{-2}
169	1.7151×10^{-2}	1.3656	1.2560×10^{-2}
625	4.9871×10^{-3}	1.0778	4.6272×10^{-3}
2401	1.4358×10^{-3}	5.7447×10^{-1}	2.4994×10^{-3}
9409	3.6343×10^{-4}	2.8995×10^{-1}	1.2534×10^{-3}
37249	1.1881×10^{-4}	1.4329×10^{-1}	8.2918×10^{-4}

almost constant during uniform mesh refinement. This is due to the fact that on these meshes the a-priori $L^\infty(\Omega)$-error estimates are valid with uniform bounded constants. This is not the case for adaptive and grading refinement. Here the efficiency index

deteriorates as the meshes get refined. This behavior is most prominent for the grading refinement, cf. Table 3. Hence, the development of a computable, reliable, and efficient error estimate is subject to future research.

5.2 Example 2

Let us report about the outcome of our numerical experiments for an example with both control and state constraints. The data of this problem is given as follows.

$$
\begin{aligned}
\Omega &= (0, 1)^2, \\
y_d &= 10(1 - x_1 - x_2)^3, \\
y_a &= -0.35, \\
y_b &= 0.4, \\
u_a &= -25, \\
u_b &= 20, \\
\alpha &= 0.001.
\end{aligned}
$$

The state equation is defined as

$$
\begin{aligned}
-\Delta y + y &= u \text{ in } \Omega, \\
y &= 0 \text{ on } \Gamma.
\end{aligned}
$$

Here, the pair

$$
(\hat{y}, \hat{u}) = (0, 0)
$$

fulfills Assumption 2 with $\tau = 0.34999$.

As for example 1, we discretize states, adjoints, and controls by continuous and piecewise linear elements, which corresponds to the choice $l = 1$. The discrete control and state for a uniform triangulation consisting of 20,000 triangles can be seen in Fig. 2.
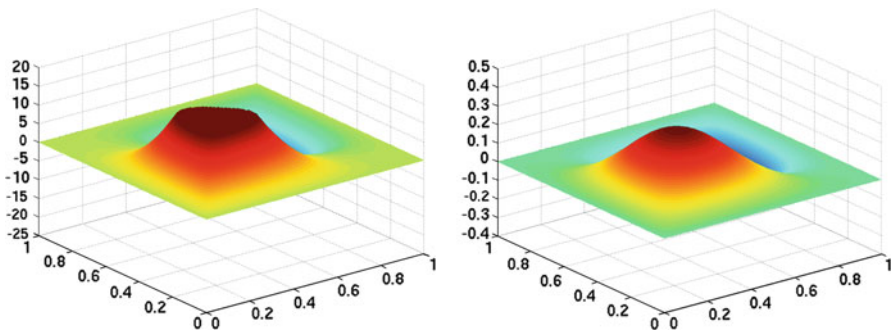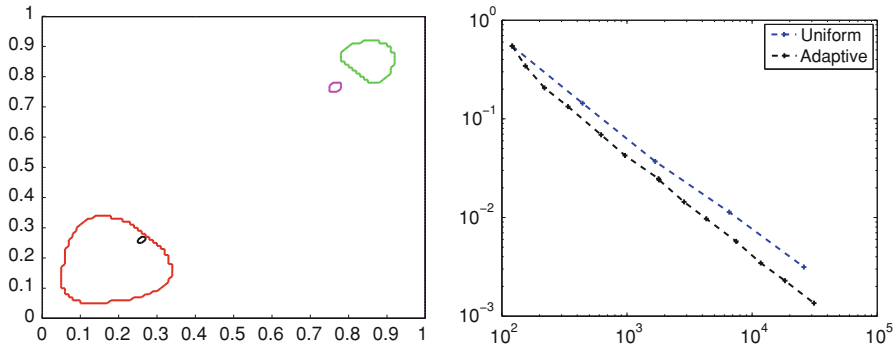

Fig. 2 Example 2: optimal discrete control and state

Fig. 3 Example 2: active sets; comparison of η versus number of unknowns for uniform and adaptive refinement

Table 4 Example 2: Results for uniform (top) and adaptive (bottom) refinement

N	η
121	5.4661×10^{-1}
441	1.4406×10^{-1}
1681	3.7044×10^{-2}
6561	1.1338×10^{-2}
25921	3.1461×10^{-3}
121	5.4661×10^{-1}
154	3.4434×10^{-1}
219	2.0604×10^{-1}
338	1.3322×10^{-1}
621	6.9029×10^{-2}
964	4.2551×10^{-2}
1787	2.4736×10^{-2}
1803	2.4068×10^{-2}
2857	1.4400×10^{-2}
4305	9.7050×10^{-3}
7455	5.7234×10^{-3}
11723	3.4542×10^{-3}
18302	2.2890×10^{-3}
31233	1.3545×10^{-3}

The boundaries of the resulting active sets are plotted in Fig. 3. The upper control and state constraints are active in the lower left part of Ω, the active sets for the lower constraints are located in the upper right part. The small sets correspond to active state constraints, the large sets to the control constraints. As one can see from the picture, the active sets for the upper state constraint is included in the active set for the upper control constraint.

Starting with a coarse grid consisting of 20 triangles, we performed numerical computations with uniform and adaptive refinement. Since the solution of the continuous problem is unknown, we show the values of the error estimator only. The evolution of the error estimator η versus the number of degrees of freedom is plotted in Fig. 3. For comparison, the precise values of η for uniform and adaptive refinement are shown in Table 4. As can be seen from these numbers, the adaptive refinement outperforms the uniform refinement strategy. With uniform refinement one needs approximately three times as much degrees of freedom to achieve the same accuracy as the adaptive procedure.

6 Conclusion

In this article, we developed a fully computable a-posteriori error estimator for state and control constrained optimal control problems. Moreover, we showed that the estimator tends to zero if the solution of the discretized problems converge to the solution of the undiscretized problem.

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